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The vibration of rectangular orthotropic plate with massive inclusions

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The problem on proper and forced vibrations of the loosely leant rectangular orthotropic plate with massive circular inclusion is considered in the paper. The flexure of the plate is described by modified equations of Timoshenko's theory of plates. Numerical solution of the problem is found by the indirect method of boundary elements based on the sequential approach to constructing generalized functions and on collocation method. The problem can be generalized on the case of arbitrary located inclusion and the arbitrary number of them. The influence of the mass of the massive circular inclusion on the proper frequencies of the plate is investigated.

1. Introduction

The problem of vibration of plates with different kinds of stiffeners and the concentrated mass is rather widely considered in science. In the articles of C.G. Boay a free vibration analysis of rectangular plates carrying a concentrated mass is analysed where the deflection of the plate is postulated by a multi-term trigonometric series function and the Ritz approach and Rayleigh-energy method is applied to rectangular plates with various edge support combinations of clamp and simple support conditions [1, 2]. In the work of B.P. Shastry and G. Venkateswara Rao free vibration of plates with arbitrary oriented stiffeners are studied using high precision plate bending and stiffener elements for both simply supported and clamped boundary conditions [6]. In the work of D. Zhou and Tianjian Ji the vibratory characteristics of rectangular plates attached with continuously and uniformly distributed spring-mass in a rectangular region are studied and the Ritz–Galerkin method is used to derive the approximate solution when the spring-mass is distributed on part the plate by using the Chebyshev polynomial series to construct the admissible functions [9]. In the paper of N. Li and D.J. Gorman the problem of free vibration of simply supported rectangular plates with internal line support along diagonals is considered and here fore the solution is used the approach where the line supports are replaced by multi point support [3]. The exact approach for free vibration analysis of rectangular isotropic plates with line-concentrated mass and elastic line-support is presented in the paper of O.S. Li [4].

2. Modified equations of the Timoshenko's theory of plates

Mathematical model of the orthotropic plate of Timoshenko's type [5] that take into consideration the normal component of the inertial force is represented by the equilibrium equations

$$\frac{\partial M_{i1}}{\partial \alpha_1} + \frac{\partial M_{i2}}{\partial \alpha_2} - Q_i = -m_i \quad (i = 1, 2), \qquad \frac{\partial Q_1}{\partial \alpha_1} + \frac{\partial Q_2}{\partial \alpha_2} - 2h\delta\frac{\partial 2u}{\partial t^2} = -q, \tag{1}$$

and physical equations

$$M_{ii} = D_i\left(\frac{\partial \gamma_i}{\partial \alpha_i} + \nu_{ij}\frac{\partial \gamma_j}{\partial \alpha_j}\right),$$

$$M_{ij} = M_{ji} = D_{ij}\left(\frac{\partial \gamma_j}{\partial \alpha_i} + \nu_{ij}\frac{\partial \gamma_i}{\partial \alpha_j}\right), \tag{2}$$

$$Q_i = \Lambda_i\left(\gamma_i + \frac{\partial w}{\partial \alpha_i}\right) \qquad (i,j=1,2,\ i\neq j),$$

where

$$D_i = \frac{2h^3 E_i}{3(1-\nu_{12}\nu_{21})}, \qquad D_{12} = \frac{2h^3 E_{12}}{3}, \qquad \Lambda_i = \frac{5hG_{i3}}{3},$$

E_i, E_{12}, G_{ij}, ν_{ij} – elastic constants ($E_1\nu_{12} = E_2\nu_{21}$, $G_{12} = G_{21}$); α_1, α_2, α_3 – Cartesian coordinates ($\alpha_3 = 0$ – equation of the middle surface); $2h$ – thickness of the plate; δ – density of the material; w, γ_1, γ_2 – deflection and rotation angles of the normal to the middle surface; Q_i, M_{ij} – inner forces; q, m_i – environmental stresses.

The displacements of an arbitrary point of the plate is determined by the following formulas,

$$u_i = \gamma_i\alpha_3 \quad (i=1,2), \qquad u_3 = w. \tag{3}$$

Modified equations of the bending of the plate can be obtained from the equations (1)–(3), not taking into account rigid rotations

$$\omega_3 = \frac{\alpha_3}{2}\left(\frac{\partial \gamma_2}{\partial \alpha_1} - \frac{\partial \gamma_1}{\partial \alpha_2}\right).$$

According to [7] let's use in the expressions for M_{12}, M_{21} the function H (response on rotation) and the small parameter $\beta = \frac{1}{D_{12}}$,

$$M_{12} = 2D_{12}\frac{\partial \gamma_1}{\partial \alpha_2} + H, \qquad M_{21} = 2D_{12}\frac{\partial \gamma_2}{\partial \alpha_1} - H, \qquad \beta H = \frac{\partial \gamma_2}{\partial \alpha_1} - \frac{\partial \gamma_1}{\partial \alpha_2}. \tag{4}$$

Degenerated system of equations that corresponds to Eqs. (1), (2) taking into account the relations (4) in the case $\beta = 0$, represents the modified model of the bending of the plate. Thus the last equation (4) can be rewritten as

$$\frac{\partial \gamma_2}{\partial \alpha_1} - \frac{\partial \gamma_1}{\partial \alpha_2} = 0.$$

It is satisfied by the insertion of the potential of the angles of rigid rotations

$$\gamma = \gamma(\alpha_1, \alpha_2), \qquad \gamma_i = -\frac{\partial \gamma}{\partial \alpha_i}, \quad (i=1,2) \tag{5}$$

After the conversion of the relations (2), taking into account (5) and after substitution them in the equation of the system (1), we will obtain the following system of equations,

$$\begin{aligned}
&\Delta^4\gamma + \frac{\Lambda}{D}\Delta^2(w-\gamma) = \frac{1}{D}\left(\frac{\partial m_1}{\partial \alpha_1} + \frac{\partial m_2}{\partial \alpha_2}\right),\\
&\Delta^2(w-\gamma) - \frac{2h}{\Lambda}\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\Lambda}q,\\
&\frac{1}{D}\Delta_0^2 H - \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2}\left[(d_1 - \nu_{21}d_2 - 2d_{12})\frac{\partial^2\gamma}{\partial \alpha_1^2} + (d_2 - \nu_{12}d_1 - 2d_{12})\frac{\partial^2\gamma}{\partial \alpha_2^2} + (b_1 - b_2)\frac{\partial^2(w-\gamma)}{\partial \alpha_1 \partial \alpha_2}\right]\\
&\qquad = -\frac{1}{D}\left(\frac{\partial m_1}{\partial \alpha_2} - \frac{\partial m_2}{\partial \alpha_1}\right),
\end{aligned} \tag{6}$$

where

$$\Delta^4 = d_1\frac{\partial^4}{\partial\alpha_1^4} + 2d_{12}\frac{\partial^4}{\partial\alpha_1^2\alpha_2} + d_2\frac{\partial^4}{\partial\alpha_2^4}, \quad \Delta^2 = b_1\frac{\partial^2}{\partial\alpha_1^2} + b_2\frac{\partial^2}{\partial\alpha_2^2}, \quad \Delta_0^2 = \frac{\partial^2}{\partial\alpha_1^2} + \frac{\partial^2}{\partial\alpha_2^2},$$

$$d_i = \frac{D_i}{D}, \quad d_{12} = \frac{D_1\nu_{12}}{D} + \frac{D_{12}}{D}, \quad b_i = \frac{\Lambda_i}{\Lambda},$$

D, Λ - normalizing factors.

Normal and tangential components of displacements and forces along the smooth curve C with the unit normal and tangential vectors $\{n_1(\alpha); n_2(\alpha)\}$, $\{\tau_1(\alpha); \tau_2(\alpha)\}$, $(\tau_1 = -n_2\,, \tau_2 = n_1)$ are determined by the formulas

$$\gamma_n = -\frac{\partial\gamma}{\partial n}, \quad \gamma_\tau = -\frac{\partial\gamma}{\partial\tau}, \qquad Q_n = -\Lambda\frac{\partial(\gamma - w)}{\partial n}$$

$$M_n = -D\left[(d_1 n_1^2 + \nu_{21}d_2 n_2^2)\frac{\partial^2\gamma}{\partial\alpha_1^2} + 4d_{12}n_1 n_2\frac{\partial^2\gamma}{\partial\alpha_1\partial\alpha_2} + (d_2 n_2^2 + \nu_{12}d_1 n_1^2)\frac{\partial^2\gamma}{\partial\alpha_2^2}\right] \tag{7}$$

$$M_\tau = -D\left\{\left[(d_1 - \nu_{21}d_2)\frac{\partial^2\gamma}{\partial\alpha_1^2} - (d_2 - \nu_{12}d_1)\frac{\partial^2\gamma}{\partial\alpha_2^2}\right]n_1 n_2 - 2d_{12}(n_1^2 - n_2^2)\frac{\partial^2\gamma}{\partial\alpha_1\partial\alpha_2}\right\} - H.$$

In the formulation of the boundary conditions in the framework of modified equations as the variables are the normal and tangential components of the moment, shearing force, deflection and normal component of the angle of rotation of the normal to the middling surface.

3. Problem statement

Let's consider the problem (without initial conditions) about the vibration of rectangular plate with massive inclusion. Exterior edge is loosely leant, the interior edge with the directing L curve (the curve of Lapunov's type) is rigidly clamped with the massive body. The body is under the impact of forces and moments with the resultants $P = P_0 \sin\theta_0 t$, $M_1 = M_{10}\sin\theta_0 t$, $M_2 = M_{20}\sin\theta_0 t$, which verify according to the harmonic law with the respect to the time co-ordinate with the frequency θ_0, Fig. 1.

Let's denote as $\Pi = \{(\alpha_1, \alpha_2) : 0 \le \alpha_1 \le l_1\,, 0 \le \alpha_2 \le l_2\}$ the rectangle with the sides l_1, l_2, the edge of it $\partial\Pi$ coincide with the exterior contour of the middle surface of the plane. On $\partial\Pi$ we have the conditions that correspond to loosely leant edge,

$$\begin{aligned} &W = 0,\ \gamma_2 = 0,\ M_{11} = 0 \quad \text{at } \alpha_1 = 0,\ \alpha_1 = l_1\,,\\ &W = 0,\ \gamma_1 = 0,\ M_{22} = 0 \quad \text{at } \alpha_2 = 0,\ \alpha_2 = l_2\,. \end{aligned} \tag{8}$$

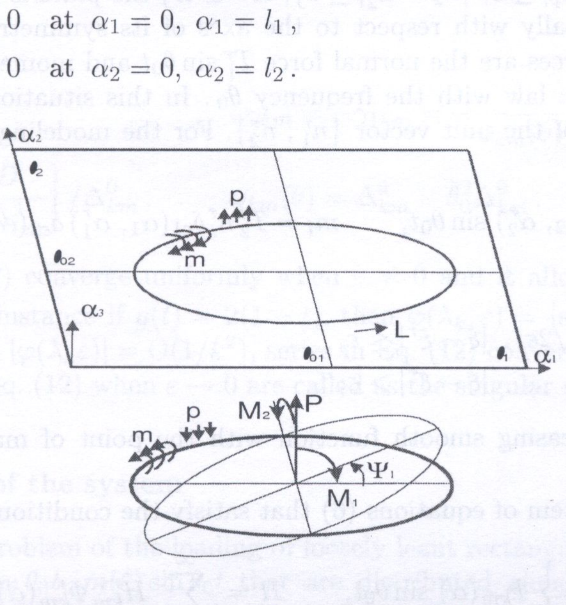

Fig. 1. The geometry of the plate and inclusion

Formulating the boundary conditions on the inner contour of the plate we suppose that the vibration of the body is according to the harmonic law with the frequency θ_0, and also we model the interaction of the body with the plate by the forces and moments $p = p_0(\xi)\sin\theta_0 t$, $m = m_0(\xi)\sin\theta_0 t$ than are distributed along the contour L. The moments are oriented in the direction of the normal to L. Let's denote as $\hat{w} = w_0 \sin\theta_0 t$, $\psi_i = \psi_{i0}\sin\theta_0 t$ $(i = 1, 2)$ the displacements of the point of localization of the resultant force and the angles of rotation of the body with respect to the axes of reduction of the resultant moments.

Thus for the deflection and normal component of the angle of rotation of the normal to the middle surface on the boundary, taking into account $\gamma_n = \frac{\partial w}{\partial n}$, we find the following expressions,

$$\begin{aligned} w(\xi,t) &= (w_0 + \xi_1\psi_{10} + \xi_2\psi_{20})\sin\theta_0 t, \\ \gamma_n(\xi,t) &= -[\psi_{10}n_1(\xi) + \psi_{20}n_2(\xi)]\sin\theta_0 t, \qquad \text{when } \xi \in L, \end{aligned} \tag{9}$$

where $\{n_1(\xi), n_2(\xi)\}$ – unit normal to the line L vector.

The forces satisfy the equation of movement

$$\begin{aligned} -I_1\frac{d^2\psi_1}{dt^2} &= M_1 + \int_L [m_0(\xi)n_1(\xi) + p_0(\xi)\xi_1]\,dl, \\ -I_2\frac{d^2\psi_2}{dt^2} &= M_2 + \int_L [m_0(\xi)n_2(\xi) + p_0(\xi)\xi_2]\,dl, \\ -\hat{m}\frac{d^2\hat{w}}{dt^2} &= -P + \int_L p_0(\xi)\,dl(\xi), \end{aligned} \tag{10}$$

where $\hat{m}$ – mass of body; I_1, I_2 – moments of inertia of the body with respect to the axes of reduction $\alpha_2 = l_{20}$, $\alpha_1 = l_{10}$.

So for the definition of the unknown functions and parameters we have the system of equations (6), (10) and the boundary conditions (8), (9).

4. Indirect method of the boundary elements

4.1. Generalized singular solution

Let's consider loosely leant plate with the middle plane in the form of rectangle Π. In the square $\Pi^r = \{\alpha(\alpha_1,\alpha_2) : |\alpha_1 - \alpha_1^r| \le \varepsilon,\ |\alpha_2 - \alpha_2^r| \le \varepsilon\}$, $\Pi^r \subset \Pi$, the plate is loaded by the forces which are distributed symmetrically with respect to the axes of its symmetry (collateral to the sides). The resultants of these forces are the normal force $T_1^r \sin\theta_0 t$ and moment $T_2^r \sin\theta_0 t$ and they vary according to the harmonic law with the frequency θ_0. In this situation the resultant moment is oriented in the direction of the unit vector $\{n_1^r, n_2^r\}$. For the modeling of this kind of impact we use delta-like functions [8]

$$q = T_1^r \delta_{\varepsilon 1}(\alpha_1, \alpha_1^r)\,\delta_{\varepsilon 2}(\alpha_2, \alpha_2^r)\sin\theta_0 t, \qquad m_i = T_2^r n_i^r \delta_{\varepsilon 1}(\alpha_1, \alpha_1^r)\,\delta_{\varepsilon 2}(\alpha_2, \alpha_2^r)\sin\theta_0 t, \tag{11}$$

where

$$\delta_\varepsilon(\xi,\xi^r) = \begin{cases} g\left(\frac{|\xi-\xi^r|}{\varepsilon}\right)/2\varepsilon, & |\xi-\xi^r| \le \varepsilon, \\ 0, & |\xi-\xi^r| > \varepsilon, \end{cases}$$

$g(\xi)$ $(0 \le \xi \le 1)$ – decreasing smooth function with the point of maximum $\xi = 0$; $g(1) = 0$; $\int_0^1 g(\xi)d\xi = 1$.

The solution of the system of equations (6) that satisfy the conditions (8) we find in the form

$$\left\{ \begin{array}{c} w \\ \gamma \end{array} \right\} = \sum_{k,m=1}^{\infty} \left\{ \begin{array}{c} w_{km}^r \\ \gamma_{km}^r \end{array} \right\} \Phi_{km}(\alpha)\sin\theta_0 t, \qquad H = \sum_{k,m=0}^{\infty} H_{km}^r \Phi_{km}^*(\alpha)\sin\theta_0 t, \tag{12}$$

where

$$\Phi_{km}(\alpha) = \sin\lambda_{1k}\alpha\sin\lambda_{2m}\alpha_2\,, \qquad \Phi^*_{km}(\alpha) = \cos\lambda_{1k}\alpha\cos\lambda_{2m}\alpha_2\,, \qquad \lambda_{1k} = \frac{k\pi}{l_1}\,, \quad \lambda_{2m} = \frac{k\pi}{l}\,.$$

Depicting the delta-like functions in Eq. (11) in the form

$$\delta_\varepsilon(\xi,\xi^r) = \frac{2}{l}\sum_{k=1}^{\infty}\varphi(\lambda_k\varepsilon)\sin\lambda_k\xi^r\sin\lambda_k\xi, \qquad \delta_\varepsilon(\xi,\xi^r) = \frac{2}{l}\left[\frac{1}{2} + \sum_{k=1}^{\infty}\varphi(\lambda_k\varepsilon)\cos\lambda_k\xi^r\cos\lambda_k\xi\right],$$

where

$$\varphi(\lambda_k\varepsilon) = \int_0^1 g(t)\cos(\lambda_k\varepsilon t)\,dt, \qquad \lambda_k = \lambda_{1k} \text{ or } \lambda_{2m}\,,$$

and substituting them with Eq. (12) in Eq. (6), we obtain the system of algebraic equations with respect to the coefficients of decisive functions. For finding the first two functions (6) we have the following system,

$$[\Delta^4_{km} - \bar\theta_0^2\Delta^0_{km}]\,w^r_{km} = \frac{C^\varepsilon_{km}}{D}\left[\Delta^0_{km}\,\Phi_{km}(\alpha^r)\,T^r_1 - \frac{\partial\Phi_{km}(\alpha^r)}{\partial n}\,T^r_2\right];$$

$$\left(\Delta^2_{km} - \frac{D\bar\theta_0^2}{\Lambda}\right)w^r_{km} - \Delta^2_{km}\gamma^r_{km} = \frac{C^\varepsilon_{km}}{\Lambda}\,\Phi_{km}(\alpha^r)\,T^r_1\,.$$

Here

$$C^\varepsilon_{km} = 4\varphi_1(\lambda_{1k}\varepsilon)\,\varphi_2(\lambda_{2m}\varepsilon)/l_1l_2\,; \qquad \bar\theta_0^2 = 2\theta_0^2h\rho/D\,;$$

$$\Delta^4_{km} = d_{11}\lambda^4_{1k} + 2d_{12}\lambda^2_{1k}\lambda^2_{2m} + d_{22}\lambda^4_{2m}\,; \qquad \Delta^0_{km} = 1 + \frac{D\Delta^4_{km}}{\Lambda\Delta^2_{km}}\,; \qquad \Delta^2_{km} = b_1\lambda^2_{1k} + b_2\lambda^2_{2m}\,.$$

The solution of the system has the form

$$\begin{aligned} w^r_{km} &= w_{km}(\alpha^r,\varepsilon) = \frac{C^\varepsilon_{km}}{D}\left[w_{1km}\,\Phi_{km}(\alpha^r)\,T^r_1 + w_{2km}\,\frac{\partial\Phi_{km}(\alpha^r)}{\partial n}\,T^r_2\right],\\ \gamma^r_{km} &= \gamma_{km}(\alpha^r,\phi) = \frac{C^\varepsilon_{km}}{D}\left[\gamma_{1km}\,\Phi_{km}(\alpha^r)\,T^r_1 + \gamma_{2km}\,\frac{\partial\Phi_{km}(\alpha^r)}{\partial n}\,T^r_2\right], \end{aligned} \tag{13}$$

where

$$w_{1km} = \frac{\Delta^0_{km}}{\omega_{km}(\theta)} \qquad w_{2km} = -\gamma_{1km} = -\frac{1}{\omega_{km}(\theta)}\,,$$

$$\gamma_{2km} = \left[\frac{1}{\omega_{km}(\theta)} + \frac{D}{\Lambda\Delta^2_{km}}\right]/\Delta^0_{km}\,, \qquad \omega_{km}(\theta) = \Delta^4_{km} - \bar\theta_0^2\Delta^0_{km}\,.$$

Binary series in Eq. (12) converge uniformly when $\varepsilon \neq 0$ and it allows us to make boundary junction when $\varepsilon \to 0$. For instance if $g(t) = 2(1-t)$, then $\varphi(\lambda_k,\varepsilon) = \left[\sin\left(\lambda_k\frac{\varepsilon}{2}\right) / \left(\lambda_k\frac{\varepsilon}{2}\right)\right]^2$ and as the result of the estimation $|\varphi(\lambda_k\varepsilon)| = \mathrm{O}(1/k^2)$, series in Eq. (12) converge uniformly.

Boundary functions in Eq. (12) when $\varepsilon \to 0$ are called as the singular solution of the system (6).

4.2. Integral equations of the system

Let's consider at first the problem of the loading of loosely leant rectangular plate by the unknown forces and moments $p(\xi)\sin\theta_0 t$, $m(\xi)\sin\theta_0 t$ that are distributed along the line L. Generalized solution of the problem is represented in the form of integral compressions of these forces and the solution of the problem of Section 3.1. For the main decisive function we have the following expressions,

$$w(\alpha,t)=\lim_{\varepsilon\to 0}\int_L\sum_{k,m=1}^{\infty}\frac{C_{km}^{\varepsilon}}{D}\left[w_{1km}\Phi_{km}(\xi)\,p(\xi)+w_{2km}m(\xi)\,\frac{\partial\Phi_{km}(\xi)}{\partial n}\right]\Phi_{km}(\alpha)\,\mathrm{d}l(\xi)\sin\theta_0 t,$$
$$\gamma(\alpha,t)=\lim_{\varepsilon\to 0}\int_L\sum_{k,m=1}^{\infty}\frac{C_{km}^{\varepsilon}}{D}\left[\gamma_{1km}\Phi_{km}(\xi)\,p(\xi)+\gamma_{2km}m(\xi)\,\frac{\partial\Phi_{km}(\xi)}{\partial n}\right]\Phi_{km}(\alpha)\,\mathrm{d}l(\xi)\sin\theta_0 t. \tag{14}$$

The expressions for the normal (with respect to the line C) component of the displacement and forces we get after substituting Eq. (14) in Eq. (7).

After the substitution of Eq. (14) in Eq. (9) and making the boundary junction when $\alpha\to\alpha_0$, $\alpha_0\in L$, (in the domain $\Pi\backslash D$ in the direction of the exterior normal to L), and taking into account Eq. (10), we get the system of three integral equations and three integral relations

$$\frac{1}{D}\int_L\begin{bmatrix} w_1(\alpha_0,\xi) & w_2(\alpha_0,\xi)\\ \gamma_{n1}(\alpha_0,\xi) & \gamma_{n2}(\alpha_0,\xi)\end{bmatrix}\begin{Bmatrix} p(\xi)\\ m(\xi)\end{Bmatrix}\mathrm{d}l(\xi)=\begin{Bmatrix} w_0+\alpha_{10}\psi_{10}+\alpha_{20}\psi_{20}\\ -[\psi_{10}n_1(\alpha_0)+\psi_{20}n_2(\alpha_0)]\end{Bmatrix},\quad \alpha_0\in L, \tag{15}$$
$$I_1\theta_0^2\psi_{10}=\int_L[\xi_1 p(\xi)+n_1(\xi)\,m(\xi)]\,\mathrm{d}l(\xi)+M_1;$$
$$I_2\theta_0^2\psi_{20}=\int_L[\xi_2 p(\xi)+n_2(\xi)\,m(\xi)]\,\mathrm{d}l(\xi)+M_2;$$
$$\hat{m}\theta_0^2 w_0=\int_L p(\xi)\,\mathrm{d}l(\xi)-P.$$

4.3. Collocation method

The approximate solution of the system of equations (15) we find by the collocation method. The line L is approached by the broken line L^*, that is formed by the rectilinear sections L^r, $r=\overline{1,N}$, across every section the unknown densities have the constant values $p(\xi)=T_1^r/2l^r$. The section L^r is defined by the length $2l^r$, the middle point $\xi^r(\xi_1^r,\xi_2^r)$ and the guide unit vector $\{\tau_1^r,\tau_2^r\}=\{\tau_1(\xi^r),\tau_2(\xi^r)\}$. The limits when $\varepsilon\to 0$ of the sums of series in Eq. (15) are approximated by the correspondent partial sums of series when $\varepsilon\neq 0$ is adequately small.

Calculating the integrals in Eq. (15) ,taking into account assumption made and minimizing the residual of the solution in the control points $\alpha^q(\alpha_1^q;\alpha_2^q)$, $q=\overline{1,N}$, – the middle points of the sections L^q, we reduce Eq. (15) to the system of algebraic equations

$$\frac{1}{D}\sum_{r=1}^{N}\begin{bmatrix} w_1^r(\alpha^q,\varepsilon) & w_2^r(\alpha^q,\varepsilon)\\ \gamma_{n1}^r(\alpha^q,\varepsilon) & \gamma_{n2}^r(\alpha^q,\varepsilon)\end{bmatrix}\begin{Bmatrix} T_1^r\\ T_2^r\end{Bmatrix}=\begin{Bmatrix} w_0+\alpha_1^q\psi_{10}+\alpha_2^q\psi_{20}\\ -[\psi_{10}n_1^q+\psi_{20}n_2^q]\end{Bmatrix},\quad q=\overline{1,N},$$
$$\sum_{r=1}^{N}\begin{bmatrix}\alpha_1^r & n_1^r\\ \alpha_2^r & n_2^r\end{bmatrix}\begin{Bmatrix} T_1^r\\ T_2^r\end{Bmatrix}=\theta_0^2\begin{Bmatrix} I_1\psi_{10}\\ I_2\psi_{20}\end{Bmatrix}-\begin{Bmatrix} M_1\\ M_2\end{Bmatrix}, \tag{16}$$
$$\sum_{i=1}^{2}\sum_{r=1}^{N}Q_{ni}^r T_i^r=\theta_0^2\hat{m}w_0+P.$$

Here, such notations are made:

$$w_i^r(\alpha,\varepsilon) = \sum_{k,m=1}^{K,M} C_{km}^{\varepsilon} w_{ikm} \Psi_{ikm}^r \Phi_{km}(\alpha),$$

$$\gamma_{ni}^r(\alpha,\varepsilon) = \sum_{k,m=1}^{K,M} C_{km}^{\varepsilon} \gamma_{ikm} \Psi_{ikm}^r \frac{\partial \Phi_{km}(\alpha)}{\partial n},$$

$$Q_{ni}^r(\alpha,\varepsilon) = \frac{\Lambda}{D} \sum_{k,m=1}^{K,M} C_{km}^{\varepsilon} (\gamma_{ikm} - w_{ikm}) \Psi_{ikm}^r \frac{\partial \Phi_{km}(\alpha)}{\partial n},$$

$$M_{ni}^r(\alpha,\varepsilon) = \sum_{k,m=1}^{K,M} C_{km}^{\varepsilon} \gamma_{ikm} \Psi_{ikm}^r \Phi_{km}^0(\alpha),$$

trigonometrical polynomial of the order K, M,

$$\Phi_{km}(\alpha) = \frac{1}{2}[\cos z_{km}(\alpha) - \cos \bar{z}_{km}(\alpha)],$$

$$\frac{\partial \Phi_{km}}{\partial n} = \frac{1}{2}[a_{km}(\alpha)\sin z_{km}(\alpha) - \bar{a}_{km}(\alpha)\sin \bar{z}_{km}(\alpha)],$$

$$\Phi_{km}^0(\alpha) = \frac{1}{2}\left\{\left[\lambda_{1k}^2 d_{n1}(\alpha) + \lambda_{1k}\lambda_{2m} d_{n12}(\alpha) + \lambda_{2m}^2 d_{n2}(\alpha)\right]\cos z_{km}(\alpha) - \left[\lambda_{1k}^2 d_{n1}(\alpha) + \lambda_{1k}\lambda_{2m} d_{n12}(\alpha) + \lambda_{2m}^2 d_{2m}(\alpha)\right]\cos \bar{z}_{km}(\alpha)\right\},$$

$$\Psi_{1km}^r = \frac{1}{2}\left[\varphi(l^r b_{km}^r)\cos z_{km}^r - \varphi(l^r \bar{b}_{km}^r)\cos \bar{z}_{km}^r\right],$$

$$\Psi_{2km}^r = \frac{1}{2}\left[a_{km}^r \varphi(l^r b_{km}^r)\sin z_{km}^r - \bar{a}_{km}^r \varphi(l^r \bar{b}_{km}^r)\sin \bar{z}_{km}^r\right],$$

$$\varphi(z) = \frac{\sin z}{z},$$

and

$$a_{km}^r = \lambda_{1k} n_1^r + \lambda_{2m} n_2^r, \qquad \bar{a}_{km}^r = \lambda_{1k} n_1^r - \lambda_{2m} n_2^r,$$

$$b_{km}^r = \lambda_{1k}\tau_1^r + \lambda_{2m}\tau_2^r, \qquad \bar{b}_{km}^r = \lambda_{1k}\tau_1^r - \lambda_{2m}\tau_2^r,$$

$$n_i^r = n_i(\alpha^r), \qquad \tau_i^r = \tau_i(\alpha^r), \qquad (n_1^r = \tau_2^r,\ n_2^r = -\tau_1^r).$$

Discrete analogs of the integral expressions for the deflection and normal to the curve C components of the angle of rotation, shearing force and moment we get from Eqs. (14) and (7) in the following form,

$$\left\{\begin{array}{c} w \\ \gamma_n \end{array}\right\} = \frac{1}{D}\sum_{r=1}^{N}\left[\begin{array}{cc} w_1^r(\alpha,\varepsilon) & w_2^r(\alpha,\varepsilon) \\ \gamma_{n1}^r(\alpha,\varepsilon) & \gamma_{n2}^r(\alpha,\varepsilon)\end{array}\right]\left\{\begin{array}{c} T_1^r \\ T_2^r \end{array}\right\}\sin\theta_0 t,$$

$$\left\{\begin{array}{c} Q_n \\ M_n \end{array}\right\} = \sum_{n=1}^{N}\left[\begin{array}{cc} Q_{n1}^r(\alpha,\varepsilon) & Q_{n2}^r(\alpha,\varepsilon) \\ M_{n1}^r(\alpha,\varepsilon) & M_{n2}^r(\alpha,\varepsilon)\end{array}\right]\left\{\begin{array}{c} T_1^r \\ T_2^r \end{array}\right\}\sin\theta_0 t, \qquad \alpha \in \Pi\backslash D, \quad \alpha \notin L^*. \tag{17}$$

Consequently, having found from Eq. (16) the values of the parameters T_i^k the normal components of displacements and forces are determined by Eqs. (17). The frequencies of vibration of the plate we can find from the condition of existence of the nontrivial solution of the homogeneous system of Eq. (16).

5. The numerical example

Let's consider the symmetric problem ($l_{01} = \frac{l_1}{2}$, $l_{02} = \frac{l_2}{2}$) of the vibration of the rectangular plate with circular inclusion with the radius R. The equation of the contour L has the form $\alpha_1 = l_{01} - R\cos t$, $\alpha_2 = l_{02} - R\sin t$, $0 \le t < 2\pi$. The contour is discriminated by N elements with the length $2l^r = 2\pi R/N$, with the centers in the points $\alpha^r(\alpha_1^r, \alpha_2^r)$ and guide vectors $\{\sin t^r, -\cos t^r\}$, where $\alpha_1^r = \frac{l_1}{2} - R\cos t^r$, $\alpha_2^r = \frac{l_2}{2} - R\sin t^r$, $t^r = \pi(2r-1)/N$, $r = \overline{1,N}$. Approximate values of the coefficients in Eq. (15) are found when $K = M = 550$, $\varepsilon/l = 0.004$ and $N = 20$.

In Fig. 2 the diagrams of the characteristic frequencies of vibrations $\overline{\theta} = l^2\theta_0\sqrt{2h\rho/D}$ of the square plate ($l_1 = l_2 = 1$, $\nu_{12} = \nu_{21} = 0.3$) depending on the relation of the mass of the inclusion to the mass of the plate element $\overline{m} = m/2h\pi R^2\rho$ are given (the line 1 corresponds to the case when $E1/E2 = 100$, line 2 - $E1/E2 = 10$ and line 3 - $E1/E2 = 1$).

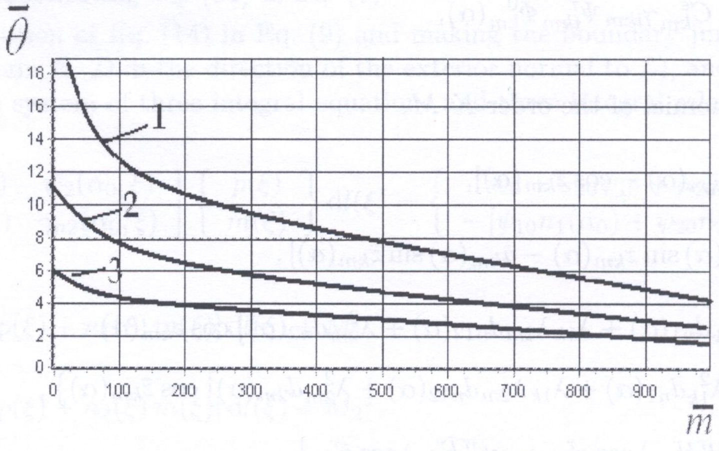

Fig. 2. Characteristic vibrations of the plate

6. Conclusion

Using the delta-like function sequences make the good convergence of the method.

As we see from the diagram when the mass of the inclusion increases the frequency of the plate decreases. The influence of the mass of the inclusion is valuable and can't be ignored.

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