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Eigenvalue analysis for high telecommunication towers with lognormal stiffness by the response function method and SFEM

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The main aim of this paper is to demonstrate the application of the generalized stochastic perturbation technique to model the lognormal random variables in structural mechanics. This is done to study probabilistic characteristics of the eigenvibrations for the high telecommunication towers with random stiffness, which are modeled as the linear elastic 3D trusses. The generalized perturbation technique based on the Taylor expansion is implemented using the Stochastic Finite Element Method in its Response Function version. The main difficulty here, in a comparison to this technique previous applications, is a necessity of both odd and even order terms inclusion in all the Taylor expansions. The hybrid numerical approach combines the traditional FEM advantages with the symbolic computing and its visualization power and it enables for a verification of probabilistic convergence of the entire computational procedure.

Keywords: stochastic dynamics, Stochastic Finite Element Method, response function method, stochastic perturbation technique

1. Introduction

The eigenvalue problems with random parameters [1, 7–10] are still important in stochastic computational mechanics, which were extensively solved before using the Monte-Carlo simulation, Karhunen-Loeve or Taylor expansions as well as some algebraic approximations. Although the stochastic perturbation technique based on the Taylor expansion was usually employed for the analysis of the Gaussian input random variables and fields [3, 4, 6], its application area is general and include all random variables types. The only condition here is that the probabilistic moments included in the specific equations for both input and output characterization must exist and must be computable. The lognormal probability distribution function (PDF), similarly to the Gaussian one, has the analytical equations describing recursively all probabilistic moments and, being popular in engineering applications, is now tested with the Generalized Stochastic Finite Element Method (GSFEM). This recursive property is very unusual, because most of the distributions has the moments generating functions, so their usage may be proposed in the numerical context only. First, we would like to test the lognormal distributions having the same first two moments as the Gaussian – it needs an extra symbolic solution to the nonlinear equations for new PDF parameters. Then, we use the Taylor series based formulas with all, odd and even, components to derive the structural response

probabilistic moments and it is done for the first few perturbation orders to notice the probabilistic convergence of the method. As we employ the generalized stochastic method, the input coefficient of variations takes the very large values (until 0.25), which is unusual for the structural applications, however it illustrates the validity of this technique within the whole range of random dispersions known from computational mechanics.

The structural application of the interest here is the telecommunication tower consisting of the steel bars with the constant-cross sectional area designed in the linear elastic range. The eigenvibrations study is necessary for this structure since the natural exposition on the stochastic wind loading, some other non-uniform and unpredictable atmospheric influences like the ice increasing the cross-sectional areas. The second, more deterministic reason, is a necessity of the reliability analysis, which is based on the normative fact that the difference between the induced vibrations frequency and eigenfrequency must be larger than 25% of this last quantity (determined in a random context below). Further computational studies will enable for a consideration of the stochastic forced vibrations.

Computational implementation is based here on the Response Function Method enabling for a reconstruction of the function relating the particular eigenfrequencies with the parameter being randomized. Numerical results of the several FEM models of this tower with the Young modulus varying around its expected value are embedded into the computer algebra system to derive the additional response functions. Then, the partial derivatives are automatically calculated and inserted into the analytical equations for any desired probabilistic moment. Once we have all the formulas given analytically, a direct inclusion of the fluctuating perturbation parameter ε as well as the input coefficient of variation α into the computations and visualization remains straightforward. Computational analysis obeys here determination of up to fourth probabilistic moments and some corresponding coefficients as the functions of this coefficient as well as of the theory order. As one may suppose, the convergence of the expected values is the same like for the Gaussian distributions but the remaining higher probabilistic moments would need decisively longer expansions than before. Further numerical and theoretical efforts will be directly connected with a determination of the reliability indices for those types of structural problems since a necessity introduced by the EU engineering codes.

2. The stochastic perturbation method for the lognormal distributions

Let us introduce the random variable $b \equiv b(\omega)$ and its probability density function as $p(b)$. Then, the expected values and the m-th central probabilistic moment are defined as

$$E[b] \equiv b^0 = \int_{-\infty}^{+\infty} bp(b)\ db, \tag{1}$$

$$\mu_m(b) = \int_{-\infty}^{+\infty} (b - E[b])^m p(b)\ db. \tag{2}$$

The basic idea of the stochastic perturbation approach is to expand all the input variables and the state functions via Taylor series about their spatial expectations using some small parameter $\varepsilon > 0$. In case of random quantity $e = e(b)$, the following expression is employed:

$$e = e^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n e}{\partial b^n} (\Delta b)^n, \tag{3}$$

where

$$\varepsilon \Delta b = \varepsilon \left(b - b^0\right) \tag{4}$$

is the first variation of b about b^0. Symbol $(.)^0$ represents the function value $(.)$ taken at the expectation b^0. Let us analyze further the expected values of any state function $f(b)$ defined analogously to the formula (3) by its expansion via Taylor series with a given small parameter ε as follows:

$$E\left[f(b),b\right]=\int_{-\infty}^{+\infty}f(b)p(b)\,db=\int_{-\infty}^{+\infty}\left(f^0+\sum_{n=1}^{\infty}\frac{1}{n!}\varepsilon^n f^{(n)}(\Delta b)^n\right)p(b)\,db. \tag{5}$$

Let us remind that this power expansion is valid only if the state function is analytic in ε and the series converge; therefore, any criteria of convergence should include the magnitude of the perturbation parameter. Perturbation parameter is taken as equal to one in numerous practical computations. Contrary to the previous analyses in this area, now the lognormal random variable is considered with the probability density function given as follows:

$$f(x)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right], \tag{6}$$

where μ and σ are its parameters. As one can easily demonstrate, the first two and k-th central probabilistic moments for b are equal now to

$$E[b]=\exp\left(\mu+\frac{\sigma^2}{2}\right), \tag{7}$$

$$\mathrm{Var}(b)=e^{\sigma^2-1}e^{2\mu+\sigma^2} \tag{8}$$

and, generally

$$\mu_k(b)=\exp\left(k\mu+\frac{k^2\sigma^2}{2}\right). \tag{9}$$

From the numerical point of view, the expansion provided in the formula (5) is carried out for the summation over the finite number of components. Now, let us focus on an analytical derivation of the probabilistic moments for the structural response function. It is easy to prove that [3]

$$\begin{aligned}E\left[f(b)\right]&=f^0(b)+\varepsilon\Delta bf^{,b}+\frac{1}{2}\varepsilon^2f^{,bb}(b)\mu_2(b)+\frac{1}{3!}\varepsilon^3f^{,bbb}(b)\mu_3(b)\\&+\frac{1}{4!}\varepsilon^4f^{,bbbb}(b)\mu_4(b)+\frac{1}{5!}\varepsilon^5f^{,bbbbb}(b)\mu_5(b)+\frac{1}{6!}\varepsilon^6f^{,bbbbbb}(b)\mu_6(b)+\ldots,\end{aligned} \tag{10}$$

where $(.)^{,b}$, $(.)^{,bb}$ denote for instance the first and the second partial derivatives with respect to b evaluated at b^0, respectively. Thanks to such an extension of the random output, any desired efficiency of the expected values as well as higher probabilistic moments can be achieved by an appropriate choice of the distribution parameters. Similar considerations lead to the fourth order expressions for a variance. The following holds

$$\begin{aligned}\mathrm{Var}(f)&=\int_{-\infty}^{+\infty}\left(f^0+\varepsilon\Delta bf^{,b}+\frac{1}{2}\varepsilon^2(\Delta b)^2f^{,bb}\right.\\&\left.+\frac{1}{3!}\varepsilon^3(\Delta b)^3f^{,bbb}+\frac{1}{4!}\varepsilon^4(\Delta b)^4f^{,bbbb}-E[f]\right)^2p\left(f(b)\right)\,db\\&=\varepsilon^2e^{(2\mu+2\sigma^2)}\left(f^{,b}\right)^2+\varepsilon^3e^{(3\mu+\frac{9}{2}\sigma^2)}f^{,b}f^{,bb}+\varepsilon^4e^{(4\mu+8\sigma^2)}\left[\frac{1}{3}f^{,b}f^{,bbb}+\frac{1}{4}(f^{,bb})^2\right].\end{aligned} \tag{11}$$

The third order probabilistic moments are derived including the lowest orders only as

$$\begin{aligned}\mu_3\left(f(b),b\right) &= \int_{-\infty}^{+\infty}\left(f(b)-E[f(b)]\right)^3 p(b)\,db \\ &= \int_{-\infty}^{+\infty}\left(f^0+\varepsilon f^{,b}\Delta b+\frac{1}{2}\varepsilon^2 f^{,bb}(\Delta b)^2+\frac{1}{3!}\varepsilon^3 f^{,bbb}(\Delta b)^3+\ldots-E[f(b)]\right)^3 p(b)\,db \\ &= \varepsilon^3\mu_3(b)\left(f^{,b}\right)^3+\frac{3}{2}\varepsilon^4\mu_4(b)\left(f^{,b}\right)^2 f^{,bb}+\frac{3}{8}\varepsilon^5\mu_5(b)\left(f^{,b}\left(f^{,bb}\right)^2+\left(f^{,b}\right)^2 f^{,bbb}\right).\end{aligned} \tag{12}$$

Finally, the fourth probabilistic moment is approximated with the first few perturbation terms as

$$\begin{aligned}\mu_4(f(b),b) &= \int_{-\infty}^{+\infty}\left(f(b)-E[f(b)]\right)^4 p(b)\,db \\ &= \varepsilon^4\mu_4(b)\left(f^{,b}\right)^4+2\varepsilon^5\mu_5(b)\left(f^{,b}\right)^3 f^{,bb}+\frac{3}{2}\varepsilon^6\mu_6(b)\left(f^{,b}\right)^2\left(f^{,bb}\right)^2.\end{aligned} \tag{13}$$

Let us mention that it is necessary to multiply in each of these equations by the relevant order probabilistic moments of the input random variables to get the algebraic form convenient for any symbolic computations. Therefore, this method in its generalized form is convenient for all the random distributions, where the above mentioned moments may be analytically derived (or at least computed for a specific combination of those distributions parameters).

3. Variational formulation

Let us consider the following set of partial differential equations adequate to the linear elastodynamic problem consisting of [2, 5]

- the equations of motion

$$\boldsymbol{D}^T\boldsymbol{\sigma}+\hat{\boldsymbol{f}}=\rho\ddot{\boldsymbol{u}}, \qquad \boldsymbol{x}\in\Omega, \qquad \tau\in[t_0,\infty), \tag{14}$$

- the constitutive equations

$$\boldsymbol{\sigma}=\boldsymbol{C}\boldsymbol{\varepsilon}, \qquad \boldsymbol{x}\in\Omega, \qquad \tau\in[t_0,\infty), \tag{15}$$

- the geometric equations

$$\boldsymbol{\varepsilon}=\boldsymbol{D}\boldsymbol{u}, \qquad \boldsymbol{x}\in\Omega, \qquad \tau\in[t_0,\infty), \tag{16}$$

- the displacement boundary conditions

$$\boldsymbol{u}=\hat{\boldsymbol{u}}, \qquad \boldsymbol{x}\in\partial\Omega_u, \qquad \tau\in[t_0,\infty), \tag{17}$$

- the stress boundary conditions

$$\boldsymbol{N}\boldsymbol{\sigma}=\hat{\boldsymbol{t}}, \qquad \boldsymbol{x}\in\partial\Omega_\sigma, \qquad \tau\in[t_0,\infty), \tag{18}$$

- the initial conditions

$$\boldsymbol{u}=\hat{\boldsymbol{u}}^0, \qquad \dot{\boldsymbol{u}}=\hat{\dot{\boldsymbol{u}}}^0, \qquad \tau\in[t_0,\infty). \tag{19}$$

It is assumed that all the state functions appearing in this system are sufficiently smooth functions of the independent variables $\boldsymbol{x}$ and τ. Let us consider the variation of $\boldsymbol{u}(\boldsymbol{x},\tau)$ in some time moment $\tau = t$, denoted by $\delta\boldsymbol{u}(\boldsymbol{x},\tau)$. Using the above equations one can show that [5]

$$-\int_{\Omega}(\boldsymbol{D}^T\boldsymbol{\sigma}+\hat{\boldsymbol{f}}-\rho\ddot{\boldsymbol{u}})^T\delta\boldsymbol{u}\,d\Omega+\int_{\partial\Omega_\sigma}(N\sigma-\hat{\boldsymbol{t}})^T\delta u\,d(\partial\Omega)=0. \tag{20}$$

Assuming further that the displacement function $\boldsymbol{u}(\boldsymbol{x},t)$ has known values at the initial moment $\boldsymbol{u}(\boldsymbol{x},t_1)=0$ and at the end of the process $\boldsymbol{u}(\boldsymbol{x},t_2)=0$, the variations of this function also equal zero at those time moments

$$\delta\boldsymbol{u}(\boldsymbol{x},t_1)=0, \qquad \delta\boldsymbol{u}(\boldsymbol{x},t_2)=0. \tag{21}$$

Integrating by parts with respect to the variables x and τ we can obtain that

$$\int_{t_1}^{t_2}\left[\delta T-\int_{\Omega}\boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon}\,d\Omega+\int_{\Omega}\hat{\boldsymbol{f}}^T\delta\boldsymbol{u}\,d\Omega+\int_{\partial\Omega}\hat{\boldsymbol{t}}^T\delta\boldsymbol{u}\,d(\partial\Omega)\right]d\tau=0, \tag{22}$$

where the kinetic energy of the region Ω is defined as [5]

$$T=\frac{1}{2}\int\rho\dot{\boldsymbol{u}}^T\dot{\boldsymbol{u}}\,d\Omega. \tag{23}$$

We also notice that

$$\delta\boldsymbol{\varepsilon}=\boldsymbol{D}\delta\boldsymbol{u}, \qquad \boldsymbol{x}\in\Omega, \qquad \tau\in[t_0,\infty). \tag{24}$$

Next, we introduce the assumption that the mass forces $\hat{\boldsymbol{f}}$ and the surface loadings $\hat{\boldsymbol{t}}$ are independent from the displacement vector $\boldsymbol{u}$, which means that the external loadings do not follow the changes in the domain initial configuration. Therefore, Eq. (22) can be modified to the following statement:

$$\delta\int_{t_1}^{t_2}(T-J_p)\,d\tau=0, \tag{25}$$

where J_p means the potential energy stored in the entire domain Ω

$$J_p=U-\int_{\Omega}\hat{\boldsymbol{f}}^T\boldsymbol{u}\,d\Omega-\int_{\partial\Omega_\sigma}\hat{\boldsymbol{t}}^T\boldsymbol{u}\,d(\partial\Omega)=0, \tag{26}$$

whereas the variation is determined with respect to the displacement function and U is the elastic strain energy given by the formula

$$U=\frac{1}{2}\int_{\Omega}\boldsymbol{\varepsilon}^T\boldsymbol{C}\boldsymbol{\varepsilon}\,d\Omega. \tag{27}$$

It is well known that Eq. (26) represents the Hamilton principle widely used in structural dynamics in conjunction with the Finite Element Method approach.

4. Computational implementation

4.1. The Response Function Method in elastodynamics

Let us consider a discretization of the displacement field $u(x,\tau)$ using the following forms [2, 5]:

$$\mathbf{u}^{\alpha}_{3\times 1}(\mathbf{x},\tau) \cong \varphi_{3\times N_{(e)}}(\mathbf{x})\mathbf{q}^{\alpha}_{N_{(e)}\times 1}(\tau), \qquad \mathbf{u}^{\alpha}_{3\times 1}(\mathbf{x},\tau) \cong \mathbf{\Phi}_{3\times N}(\mathbf{x})\mathbf{r}^{\alpha}_{N\times 1}(\tau), \tag{28}$$

where $\mathbf{q}$ is a vector of the generalized coordinates for the considered finite element, $\mathbf{r}$ is a vector for the generalized coordinates of the entire discretized system, $N_{(e)}$ is the total number of the e-th finite element degrees of freedom, N is the total number of degrees of freedom in the structure model. The generalized coordinates vector for the entire structure model is composed from the finite element degrees of freedom and the transformation matrix as

$$\mathbf{r}^{\alpha}_{N\times 1} = \mathbf{a}_{N\times N_{(e)}}\mathbf{q}^{\alpha}_{N_{(e)}\times 1}, \tag{29}$$

φ and Φ are the corresponding shape function matrices (local and global). Contrary to the classical formulations of both FEM and the perturbation-based Stochastic Finite Element Method we introduce here the additional index $\alpha = 1,\ldots,M$ to distinguish between various solutions of the elastodynamic problem necessary to build up the response function (around the mean value of the input random parameter). The strain tensor can be expressed as

$$\boldsymbol{\varepsilon}^{\alpha}_{6\times 1}(\mathbf{x},\tau) = \mathbf{B}_{6\times N_{(e)}}(\mathbf{x})\mathbf{q}^{\alpha}_{N_{(e)}\times 1}(\tau) = \tilde{\mathbf{B}}_{6\times N}(\mathbf{x})\mathbf{r}^{\alpha}_{N\times 1}(\tau). \tag{30}$$

Let us denote by E the total number of finite elements in the model. Then, the Hamilton principle is obtained as

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}\sum_{e=1}^{E}(\mathbf{q}^{\alpha})^T\,\mathbf{m}^{\alpha}_{N_{(e)}\times N_{(e)}}\mathbf{q}^{\alpha} - \frac{1}{2}\sum_{e=1}^{E}(\mathbf{q}^{\alpha})^T\,\mathbf{k}^{\alpha}_{N_{(e)}\times N_{(e)}}\mathbf{q}^{\alpha} + \sum_{e=1}^{E}\left(\mathbf{Q}^{\alpha}_{N_{(e)}}\right)^T\mathbf{q}^{\alpha}\right)d\tau = 0, \tag{31}$$

so that

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}(\dot{\mathbf{r}}^{\alpha})^T\,\mathbf{M}^{\alpha}\dot{\mathbf{r}}^{\alpha} - \frac{1}{2}(\mathbf{r}^{\alpha})^T\,\mathbf{K}^{\alpha}\mathbf{r}^{\alpha} + (\mathbf{R}^{\alpha})^T\,\mathbf{r}^{\alpha}\right)d\tau = 0. \tag{32}$$

The element and global mass matrices are defined as

$$\mathbf{m}^{\alpha}_{N_{(e)}\times N_{(e)}} = \int\limits_{\Omega_e}\rho^{\alpha}(\mathbf{x})\left(\mathbf{B}_{N_{(e)}\times 6}(\mathbf{x})\right)^T\mathbf{B}_{6\times N_{(e)}}(\mathbf{x})\,d\Omega \tag{33}$$

and

$$\mathbf{M}^{\alpha}_{N\times N} = \int\limits_{\Omega}\rho^{\alpha}(\mathbf{x})\left(\widetilde{\mathbf{B}}_{N\times 6}(\mathbf{x})\right)^T\widetilde{\mathbf{B}}_{6\times N}(\mathbf{x})\,d\Omega. \tag{34}$$

The stiffness matrices at the element and at the global scales are defined as follows

$$\mathbf{k}^{\alpha}_{N_{(e)}\times N_{(e)}} = \int\limits_{\Omega_{(e)}}\left(\mathbf{B}_{N_{(e)}\times 6}\right)^T\mathbf{C}^{\alpha}_{6\times 6}\mathbf{B}_{6\times N_{(e)}}\,d\Omega \tag{35}$$

and

$$\mathbf{K}^{\alpha}_{N\times N} = \int\limits_{\Omega_{(e)}}\left(\widetilde{\mathbf{B}}_{N\times 6}\right)^T\mathbf{C}^{\alpha}_{6\times 6}\widetilde{\mathbf{B}}_{6\times N}\,d\Omega. \tag{36}$$

Hence, equation (32) can be rewritten with those substitutions as

$$(\dot{\mathbf{r}}^{\alpha})^T \mathbf{M}^{\alpha} \delta \mathbf{r} - \int_{t_1}^{t_2} \left((\ddot{\mathbf{r}}^{\alpha})^T \mathbf{M}^{\alpha} + (\mathbf{r}^{\alpha})^T \mathbf{K}^{\alpha} - (\mathbf{R}^{\alpha})^T \right) \delta \mathbf{r} \, d\tau = 0. \tag{37}$$

Considering the assumptions

$$\delta \mathbf{r}(t_1) = 0, \qquad \delta \mathbf{r}(t_2) = 0, \tag{38}$$

we finally obtain the dynamic equilibrium system

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha}, \tag{39}$$

which represents the equations of motion of the discretized system. We complete this equation with the component $\mathbf{C}^{\alpha}_{N\times N} \mathbf{r}^{\alpha}_{N\times 1}$ getting

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{C}^{\alpha} \dot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha}. \tag{40}$$

Then we decompose the damping matrix as [6]

$$\mathbf{C}^{\alpha} = \alpha_0 \mathbf{M}^{\alpha} + \alpha_1 \mathbf{K}^{\alpha}, \tag{41}$$

where the coefficients α_0 and α_1 are determined using the specific eigenfunctions for this problem, so that

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \alpha_0 \mathbf{M}^{\alpha} \dot{\mathbf{r}}^{\alpha} + \alpha_1 \mathbf{K}^{\alpha} \dot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha}, \tag{42}$$

where no summation over the doubled indices α is applied here. As it is known, the case of undamped and free vibrations leads to the system

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{0} \tag{43}$$

and the solution $\mathbf{r}^{\alpha} = \mathbf{A}^{\alpha} \sin \omega_{\alpha} t$ leads to the relation

$$-\mathbf{M}^{\alpha} \mathbf{A}^{\alpha} \omega_{\alpha}^2 \sin \omega_{\alpha} t + \mathbf{K}^{\alpha} \mathbf{A}^{\alpha} \sin \omega_{\alpha} t = \mathbf{0}, \tag{44}$$

so that for $\sin \omega_{\alpha} t \neq 0$ and $\mathbf{A}^{\alpha} \neq \mathbf{0}$ there holds

$$-\mathbf{M}^{\alpha} \omega_{\alpha}^2 + \mathbf{K}^{\alpha} = \mathbf{0}. \tag{45}$$

When the index α is postponed, then the stochastic problem is solved in a straightforward manner analogously to the previous methods and the methodology follows the successive solutions of the increasing order equations proposed in Section 3.

4.2. The Direct Differentiation Method in the eigenvalue problem

Let us consider a deterministic eigenproblem in its matrix description for its further stochastic expansion

$$\left(\mathbf{K} - \tilde{\omega}^2 \mathbf{M} \right) \varphi = \mathbf{0}. \tag{46}$$

Its zeroth order version looks like

$$\left(\mathbf{K}^0 - \left(\tilde{\omega}^0 \right)^2 \mathbf{M}^0 \right) \varphi^0 = \mathbf{0}. \tag{47}$$

After some algebraic transformation one can get the first order equation in the following form:

$$\left(\mathbf{K}^{,b} - 2\tilde{\omega}^0\tilde{\omega}^{,b}\mathbf{M}^0 - (\tilde{\omega}^0)^2\mathbf{M}^{,b}\right)\varphi^0 = -\left(\mathbf{K}^0 - (\tilde{\omega}^0)^2\mathbf{M}^0\right)\varphi^{,b}. \tag{48}$$

The next differentiation of Eq. (47) with respect to the input random variable b returns

$$\begin{aligned}&\mathbf{K}^{,bb}\varphi^0 + 2\mathbf{K}^{,b}\varphi^{,b} + \mathbf{K}^0\varphi^{,bb} - 2\tilde{\omega}^0(\tilde{\omega}^{,b})^2\mathbf{M}^0\varphi^0 - 2\tilde{\omega}^0\tilde{\omega}^{,bb}\mathbf{M}^0\varphi^0\\ &\quad - 4\tilde{\omega}^0\tilde{\omega}^{,b}(\mathbf{M}^{,b}\phi^0 + \mathbf{M}^0\varphi^{,b}) - (\tilde{\omega}^0)^2(\mathbf{M}^{,bb}\varphi^0 + 2\mathbf{M}^{,b}\varphi^{,\mathbf{b}} + \mathbf{M}^0\varphi^{,bb}) = 0.\end{aligned} \tag{49}$$

It is quite clear here that the generalized version of the stochastic perturbation technique based on the n-th order Taylor series expansion may lead to the very complex equation corresponding to the highest order closure of the entire system. One demonstrates [4] that Eq. (46) may be rewritten as

$$\sum_{k=0}^{n}\binom{n}{k}\mathbf{K}^{(n-k)}\varphi^k = \sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k-1}\binom{k-1}{l}(2\tilde{\omega})^{(k-(l+1))}\tilde{\omega}^{(l+1)}\sum_{m=0}^{n-k}\binom{n-k}{m}\mathbf{M}^{(m)}\varphi^{(n-k-m)}. \tag{50}$$

The solution to this equation makes it possible to determine up to the n-th order eigenvalues together with the corresponding eigenvectors.

5. Numerical illustration

Computational analysis has been provided on the example of the steel telecommunication tower with the height equal to 52.0 meters presented schematically in Fig. 1.

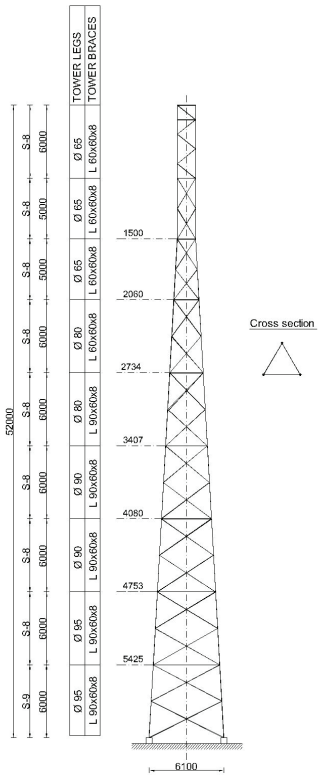

Fig. 1. Static scheme of the telecommunication tower

The entire structure has been discretized using the two-noded 183 linear space structure finite elements (3D truss elements) joined in 66 nodal points and fully supported at the ground level. All the structural members have been manufactured with the stainless steel with Young modulus equal to $E = 205\,\mathrm{GPa}$ treated here as the input lognormal random variable; analogous numerical studies with the Gaussian random variables were provided in [4]. The expected value $E[E]$ and the coefficient of variation $\alpha(E)$ of this variable were the input parameter and they were used to determine this distribution input parameters. Therefore, the following nonlinear equations system was solved symbolically using the computer algebra system MAPLE 13, to determine those coefficients

$$\begin{cases} \exp\left(\mu + \frac{\sigma^2}{2}\right) = E[E] \\ \exp\left(\sigma^2 - 1\right)\exp\left(2\mu + \sigma^2\right) = \alpha^2(E)E^2[E] \end{cases} \tag{51}$$

Since full analytical parametric solution to this system with respect to $\alpha(E)$ was impossible, the solution variability with respect to the specific values of this parameter was computed and presented in Table 1 below. The combinations of μ and σ were further used in the SFEM computations to verify the probabilistic convergence of the moments in this method with the lognormal input.

Table 1. The parameters for the input lognormal distribution

α	σ	μ
0	0	26.04627582
0,025	0.02499609497	26.04596342
0,050	0.04996879205	26.04502738
0,075	0.07489485110	26.04347120
0,100	0.09975134473	26.04130065
0,125	0.1245158079	26.03852373
0,150	0.1491663795	26.03515052
0,175	0.1736819337	26.03119311
0,200	0.1980421997	26.02666546
0,225	0.2222278668	26.02158321
0,250	0.2462206761	26.01596351
0,275	0.2700034952	26.00982488
0,300	0.2935603781	26.00318697

Numerical results based on the SFEM application obey in turn the following moments: the expected values for first (Fig. 2), third (Fig. 3) and ninth (Fig. 4) eigenvibrations frequencies and further – standard deviations (Fig. 5), third (Fig. 6) and fourth (Fig. 7) eigenvibrations frequencies.

They are all given as the functions of the input coefficient of variation; first two moments are additionally also shown as the functions of the perturbation order – from first to fourth (standard deviations) or to ninth (expectations). Let us mention that the input coefficient of variation for structural steels is usually smaller or equal to 0.1, however this range has been extended more than twice to check the entire numerical method. The particular results show that the first two moments for lower order computations are almost independent from this input coefficient and after some critical order (like the fourth one for the expected values) we notice the relationship: the higher analysis order – the larger dependence on the input coefficient of variation. A comparison of the first three diagrams leads to the conclusion that the expectations of the eigenfrequencies are not always directly proportional to the coefficient α – the third eigenvalue shows an inverse proportionality, for instance. Let us note further that for all eigenfrequencies being analyzed the probabilistic convergence is sufficient since there are no visual differences between the models based on eighth and ninth order perturbations. It should be clearly underlined that the differences between the neighbouring orders approximations are relatively larger than for the SFEM studies with the Gaussian random variables in the same random dispersion range. Figure 5 documents clearly that the

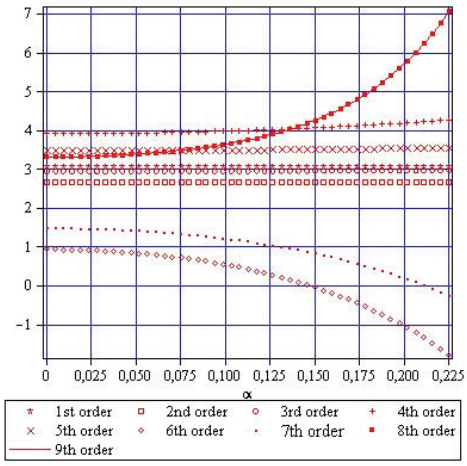

Fig. 2. The expected values for first eigenvibrations frequency

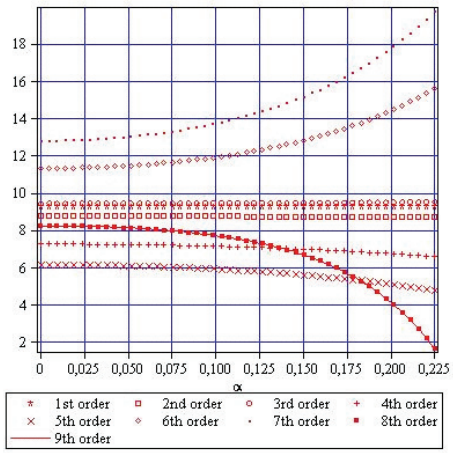

Fig. 3. The expected values for third eigenvibrations frequency

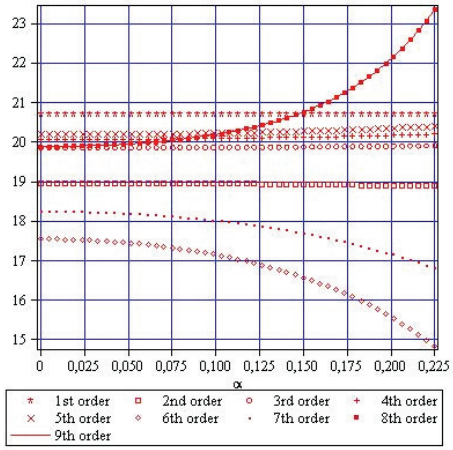

Fig. 4. The expected values for ninth eigenvibrations frequency

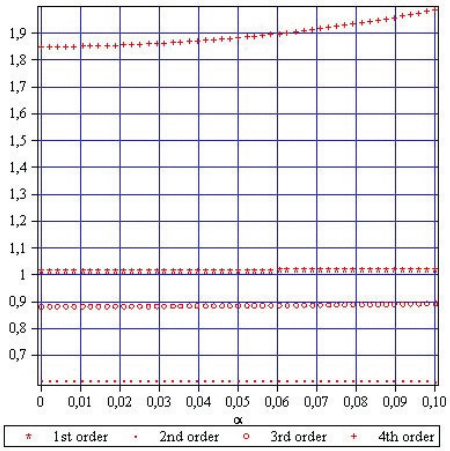

Fig. 5. The standard deviations for first eigenvibrations frequency

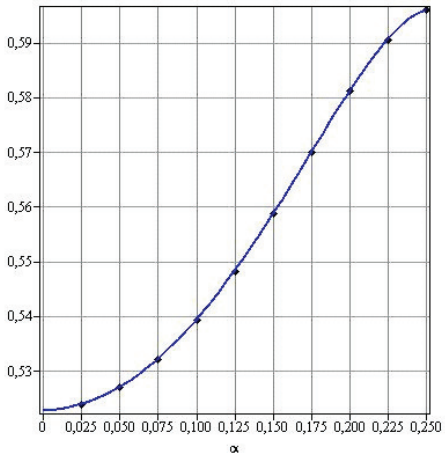

Fig. 6. Third probabilistic moments for first eigenvibrations frequency

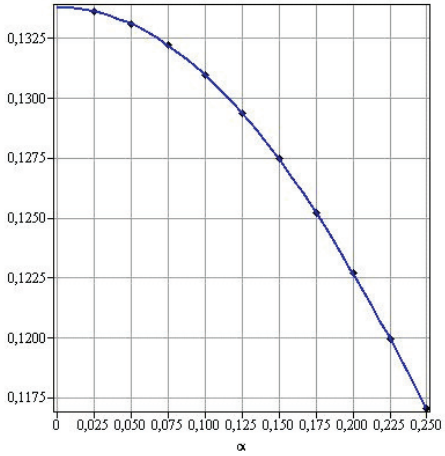

Fig. 7. Fourth probabilistic moments for first eigenvibrations frequency

fourth order analysis is decisively not sufficient for the precise estimation of the standard deviations – the reason is in full expansion with the odd and even order perturbation terms. Higher moments show similarity to the results with Gaussian input (Figs. 6 and 7) since the first few terms only may be decisive for the overall values of those moments.

6. Concluding remarks

The application of the stochastic perturbation technique based on Taylor expansion of the general order together with the additional computational implementation for lognormal variables have been shown above. Numerical analysis was provided using the Response Function Method, a traditional FEM engineering package as well as symbolic computational routines programmed in MAPLE 13. The steel telecommunication tower with the Young modulus randomized according to the lognormal distribution was studied computationally in the context of the probabilistic moments of its eigenvalues. As it is documented here, the Taylor expansions are significantly longer and much more complex than for the Gaussian distributions studied before [3, 4, 6]. Probabilistic convergence of first two probabilistic moments together with the analysis order also seems to be much slower. A comparison with the results obtained for the Gaussian distributions needs an extra solution for the nonlinear equations system, so that the hybrid symbolic-traditional FEM computational technique is very reasonable. This model has been introduced to study further the influence of the ice covers on the overall behavior of the towers and masts leading in cold regions to frequent engineering failures. The methodology may be also straightforwardly used in the stochastic reliability analyses for such structures, where the limit function may be provided as a difference between the eigenvibrations and those induced into this structure by the wind, for instance.

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