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Vibrations of a spherical shell. Comparison of 3-D elasticity and Kirchhoff shell theory results

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Natural frequencies of a vibrating hollow, elastic sphere are determined using both the 3-D elasticity and Kirchhoff shell theory.

1. INTRODUCTION

The purpose of this paper is twofold:

- Practically the only problem of fluid-structure interaction in underwater acoustic admitting a closed form solution — vibrations of an elastic spherical shell submerged in water, is solved based on the linear shell theory [3]. At the same time, the Finite Element / Boundary Element Method pursued in [2] is based on the generic elasticity formulation. As the vibrating sphere problem is used for validating the code, it would be very desirable to derive an exact solution to the coupled problem based on the plane elasticity formulation. Working out the vibrations in vacuum is just the first step in this direction.

- It has been proved in [2], using asymptotic methods, that under appropriate assumptions, solution of the elasticity problem converges to the related Kirchhoff elastic shell equations, as thickness approaches zero. The presented study can be viewed therefore as an illustration of that general result, allowing perhaps for a little more intuition and concrete physical interpretation.

The plan of the paper is as follows. Following [4] we use the Helmholtz potentials to reduce the original problem to the Helmholtz equation in a spherical domain, which is solved using the classical separation of variables approach, reviewed in Section 2.2. The transcendental eigenvalue problem presented in Section 2.3 involves spherical Bessel functions and we take this opportunity to discuss shortly a stable algorithm for their evaluations in Section 2.4.

Section 3 is devoted to a short review of the shell theory and finally the actual numerical experiments and the accompanying discussion are summarized in Section 4.

2. 3-D ELASTICITY SOLUTION

The problem of interest — natural vibrations of a spherical elastic hollow sphere in vacuo, consists of solving the steady state version of the transient linear elasticity equations accompanied by homogeneous traction boundary conditions imposed on both surfaces of the sphere. All calculations are done using the spherical coordinates

$$x = r \sin\theta \cos\phi\,, \tag{1}$$

$$y = r \sin\theta \sin\phi\,, \tag{2}$$

$$z = r \cos\theta\,. \tag{3}$$

First, by means of the Helmholtz potentials approach, the original equations are replaced with an equivalent system of those decoupled Helmholtz equations to be solved for the Helmholtz potentials. Each of the Helmholtz equations is then solved using the classical separation of variables method. Finally, imposition of traction boundary conditions leads to a series of 4 by 4 transcendental eigenvalue problems in terms of appropriate coefficients in the final series representation.

Numerical solution of those problems involves evaluation of spherical Bessel functions and a stable algorithm for their evaluation is reviewed in the last Subsection.

2.1. Helmholtz potential

The motion of an isotropic, homogeneous elastic body is governed by Navier's equations

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{4}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the unknown displacement field, $\mathbf{f}$ are prescribed body forces, ρ is the density and λ and μ are Lame's constants. Assuming that body force $\mathbf{f}$ is sufficiently smooth, we can represent them in the form.

$$\mathbf{f} = \nabla f + \nabla\times\mathbf{F} \tag{5}$$

where f and $\mathbf{F}$ are scalar and vector potentials respectively. Assuming the same form for the solution

$$\mathbf{u} = \nabla\Phi + \nabla\times\boldsymbol{\Psi} \tag{6}$$

with Φ and $\boldsymbol{\Psi}$ being the unknown scalar and vector potentials, we substitute (6) into (4) to obtain

$$\nabla\left(c_1^2\nabla^2\Phi + f - \frac{\partial^2\Phi}{\partial t^2}\right) + \nabla\times\left(c_2^2\nabla^2\boldsymbol{\Psi} + \mathbf{F} - \frac{\partial^2\boldsymbol{\Psi}}{\partial t^2}\right) = 0\,. \tag{7}$$

Here, $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$ and $c_2 = \sqrt{\frac{\mu}{\rho}}$ are longitudinal wave velocity and shear wave velocity, respectively.

If $c_1^2\nabla^2\Phi + f - \frac{\partial^2\Phi}{\partial t^2} = 0$ and $c_2^2\nabla^2\boldsymbol{\Psi} + \mathbf{F} - \frac{\partial^2\boldsymbol{\Psi}}{\partial t^2} = 0$ then Navier's equations (4) are obviously satisfied. The nontrivial question whether every solution of Navier's equation admits a representation (6) was answered positively in the completeness proof provided by Long [5].

For spherical coordinates, we additionally represent the vector potential $\boldsymbol{\Psi}$ [4] in the form

$$\boldsymbol{\Psi} = r\Psi\mathbf{e_r} + l\nabla\times(r\chi\mathbf{e_r}) \tag{8}$$

where l is a length factor in order to make the dimension of two terms in (8) the same and Ψ and χ are two unknown scalar-valued functions.

With body forces neglected, this leads to a final system of three decoupled wave equations to be solved for potentials Φ, Ψ and χ

$$c_1^2\nabla^2\Phi = \ddot{\Phi}\,, \tag{9}$$

$$c_2^2\nabla^2\Psi = \ddot{\Psi}\,, \tag{10}$$

$$c_2^2\nabla^2\chi = \ddot{\chi}\,. \tag{11}$$

Substituting (8) into (6), we can obtain the displacement field

$$u_r = \frac{\partial \Phi}{\partial r} + l\left[\frac{\partial^2 (r\chi)}{\partial r^2} - r\nabla^2\chi\right], \tag{12}$$

$$u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \phi} + \frac{l}{r}\frac{\partial^2 (r\chi)}{\partial\theta\partial r}, \tag{13}$$

$$u_\phi = \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi} - \frac{\partial \Psi}{\partial \theta} + \frac{l}{r\sin\theta}\frac{\partial^2 (r\chi)}{\partial\phi\partial r}. \tag{14}$$

From the displacement-strain relationships and the constitutive equations of linear elastic isotropic material, we can get the following stress field:

$$\sigma_{rr} = \lambda\nabla^2\Phi + 2\mu\frac{\partial^2\Phi}{\partial r^2} + 2\mu l\frac{\partial}{\partial r}\left[\frac{\partial^2 (r\chi)}{\partial r^2} - r\nabla^2\chi\right], \tag{15}$$

$$\begin{aligned}\sigma_{\theta\theta} &= \lambda\nabla^2\Phi + 2\mu\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) + \frac{2\mu}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\phi}\right) \\ &\quad + 2\mu l\left[\frac{1}{r^2}\frac{\partial^3(r\chi)}{\partial\theta^2\partial r} + \frac{1}{r}\left(\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right)\right],\end{aligned} \tag{16}$$

$$\begin{aligned}\sigma_{\phi\phi} &= \lambda\nabla^2\Phi + 2\mu\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\cot\theta\frac{\partial\Phi}{\partial\theta}\right] + \frac{2\mu}{r\sin\theta}\left(\cot\theta\frac{\partial\Psi}{\partial\phi} - \frac{\partial^2\Psi}{\partial\phi\partial\theta}\right) \\ &\quad + 2\mu l\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^3(r\chi)}{\partial\phi^2\partial r} + \frac{1}{r}\left(\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right) + \frac{\cot\theta}{r^2}\frac{\partial^2(r\chi)}{\partial\theta\partial r}\right],\end{aligned} \tag{17}$$

$$\begin{aligned}\sigma_{r\theta} &= \frac{2\mu}{r}\left(\frac{\partial^2\Phi}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) - \frac{\mu}{r\sin\theta}\left(\frac{\partial\Psi}{\partial\phi} - r\frac{\partial^2\Psi}{\partial r\partial\phi}\right) \\ &\quad + \frac{\mu l}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right) - \frac{1}{r}\frac{\partial^2(r\chi)}{\partial\theta\partial r} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2(r\chi)}{\partial\theta\partial r}\right)\right],\end{aligned} \tag{18}$$

$$\begin{aligned}\sigma_{r\phi} &= \frac{2\mu}{r\sin\theta}\left(\frac{\partial^2\Phi}{\partial r\partial\phi} - \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right) + \frac{\mu}{r}\left[2\frac{\partial\Psi}{\partial\theta} - \frac{\partial^2(r\Psi)}{\partial\theta\partial r}\right] \\ &\quad + \frac{\mu l}{r\sin\theta}\left[\frac{\partial}{\partial\phi}\left(2\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right) - \frac{2}{r}\frac{\partial^2(r\chi)}{\partial\phi\partial r}\right],\end{aligned} \tag{19}$$

$$\begin{aligned}\sigma_{\theta\phi} &= \frac{2\mu}{r^2\sin\theta}\left(\frac{\partial^2\Phi}{\partial\theta\partial\phi} - \cot\theta\frac{\partial\Phi}{\partial\phi}\right) + \frac{\mu}{r}\left(\cot\theta\frac{\partial\Psi}{\partial\theta} - \frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right) \\ &\quad + \frac{2\mu l}{r^2\sin\theta}\left[\frac{\partial^3(r\chi)}{\partial r\partial\theta\partial\phi} - \cot\theta\frac{\partial^2(r\chi)}{\partial r\partial\phi}\right].\end{aligned} \tag{20}$$

2.2. Separation of variables for the Helmholtz operator in a spherical domain

By means of the separation of variables, solution of the Helmholtz equation in spherical coordinates (r, θ, ϕ) is reduced to solving three independent ordinary differential equations: the Bessel equation in r, the Legendre equation in θ and a simple, second order equation in ϕ.

The Helmholtz equation

$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{21}$$

takes the following form in the spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=0\,. \tag{22}$$

Assuming the harmonic variation in time, we postulate the following form of the solution

$$f(r,\theta,\phi,t) = F(r,\theta,\phi)e^{-i\omega t} = F_1(r)F_2(\theta)F_3(\phi)e^{-i\omega t}\,. \tag{23}$$

Substituting (23) into (22) and following the usual reasoning we get

$$r^2\frac{\mathrm{d}^2F_1}{\mathrm{d}r^2}+2r\frac{\mathrm{d}F_1}{\mathrm{d}r}+\left(k^2r^2-p^2\right)F_1 = 0\,, \tag{24}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}F_2}{\mathrm{d}\theta}\right)+\left(p^2-\frac{q^2}{\sin^2\theta}\right)F_2 = 0\,, \tag{25}$$

$$\frac{\mathrm{d}^2F_3}{\mathrm{d}\phi^2}+q^2F_3 = 0\,, \tag{26}$$

where $k=\omega/c$ is the wave number and p^2, q^2 are separation constants. Note that p^2 and q^2 are here real numbers. If we change the variables with $p^2=\nu(\nu+1)$, $\mu=\cos\theta$ and $F_1=(kr)^{-\frac{1}{2}}R(r)$ then Eqs. (24) and (25) are transformed into the Bessel's equation

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+r\frac{\mathrm{d}R}{\mathrm{d}r}+\left[k^2r^2-\left(\nu+\frac{1}{2}\right)^2\right]R = 0 \tag{27}$$

and the Legendre's equation

$$\left(1-\mu^2\right)\frac{\mathrm{d}^2F_2}{\mathrm{d}\mu^2}-2\mu\frac{\mathrm{d}F_2}{\mathrm{d}\mu}+\left[\nu(\nu+1)-\frac{m^2}{1-\mu^2}\right]F_2 = 0\,. \tag{28}$$

Since F_3 must be a single valued periodic function with period 2π, separation constant q must be an integer, say m. For the special case q=0, solution of (26) reduces to a constant function. For q $\neq 0$, solution F_3 is a linear combination of $e^{iq\phi}$ and $e^{-iq\phi}$. Next, applying the Frobenius methods to solve the Legendre's equation we find out that a necessary and sufficient condition for the solution to exist is that ν must be a non-negative integer. The corresponding solution is then the classical associated Legendre function.

Summarizing, we get the final solution in the form

$$f(r,\theta,\phi,t) = \frac{1}{(kr)^{\frac{1}{2}}}\left[AJ_{n+\frac{1}{2}}(kr)+BY_{n+\frac{1}{2}}(kr)\right][CP_n^m(\mu)+DQ_n^m(\mu)][Ee^{im\phi}+Fe^{-im\phi}]e^{-i\omega t} \tag{29}$$

where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are the associated Legendre functions $J_{n+\frac{1}{2}}(kr)$ is the Bessel function of the first kind and $n+\frac{1}{2}$ order and $Y_{n+\frac{1}{2}}(kr)$ is the Bessel function of the second kind and $n+\frac{1}{2}$ order. The associated Legendre function Q_n^m is singular at $\mu=\pm1$, therefore we exclude it from the solution of the sphere problem. We can express then the general Helmholtz potentials as

$$\Phi = Z_n^{(i)}(\alpha r)P_n^m(\cos\theta)\exp[i(m\phi-\omega t)]\,, \tag{30}$$

$$\Psi = Z_n^{(i)}(\beta r)P_n^m(\cos\theta)\exp[i(m\phi-\omega t)]\,, \tag{31}$$

$$\chi = Z_n^{(i)}(\beta r)P_n^m(\cos\theta)\exp[i(m\phi-\omega t)]\,, \tag{32}$$

where

$$\alpha\equiv\omega/c_1,\qquad \beta\equiv\omega/c_2, \tag{33}$$

$$Z_n^{(1)} \equiv j_n(kr) \equiv (\pi/2kr)^{1/2}J_{n+\frac{1}{2}}(kr)\,, \tag{34}$$

$$Z_n^{(2)} \equiv y_n(kr) \equiv (\pi/2kr)^{1/2}Y_{n+\frac{1}{2}}(kr)\,, \tag{35}$$

and $j_n(kr)$ and $y_n(kr)$ are spherical Bessel functions.

The displacement field is derived by substituting Eqs. (30)–(32) into Eqs. (12)–(14)

$$u_r = \frac{1}{r}\left[U_1^{(i)}(\alpha r) + lU_3^{(i)}(\beta r)\right] P_n^m(\cos\theta)\exp[i(m\phi-\omega t)]\,, \tag{36}$$

$$\begin{aligned} u_\theta = \frac{1}{r}\Big\{ & V_1^{(i)}(\alpha r)\left[n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta)\right] + V_2^{(i)}(\beta r)\frac{im}{\sin\theta}P_n^m(\cos\theta) \\ & + lV_3^{(i)}(\beta r)\left[n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta)\right]\Big\}\exp[i(m\phi-\omega t)]\,, \end{aligned} \tag{37}$$

$$\begin{aligned} u_\phi = \frac{1}{r}\Big\{ & V_1^{(i)}(\alpha r)\frac{im}{\sin\theta}P_n^m(\cos\theta) - rV_2^{(i)}(\beta r)\left[n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta)\right] \\ & + lV_3^{(i)}(\beta r)\frac{im}{\sin\theta}P_n^m(\cos\theta)\Big\}\exp[i(m\phi-\omega t)]\,, \end{aligned} \tag{38}$$

where

$$U_1^{(i)}(\alpha r) = nZ_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r)\,, \qquad U_3^{(i)}(\beta r) = n(n+1)Z_n^{(i)}(\beta r)\,, \tag{39}$$

$$V_1^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r)\,, \qquad V_2^{(i)}(\beta r) = Z_n^{(i)}(\beta r)\,, \tag{40}$$

$$V_3^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\,. \tag{41}$$

$U_1^{(i)}$, $V_1^{(i)}$, $W_1^{(i)}=V_1^{(i)}$ correspond to the function Φ, $U_2^{(i)}=0$, $V_2^{(i)}$, $W_2^{(i)}=V_2^{(i)}$ correspond to the function Ψ and $U_3^{(i)}$, $V_3^{(i)}$, $W_3^{(i)}=V_3^{(i)}$ correspond to the function χ.

Using the usual displacement-strain relations and the constitutive equations of a linear isotropic elastic material, we get the following stress field represented in terms of scalar functions Φ, Ψ and χ:

$$\sigma_{rr} = \frac{2\mu}{r^2}\left[T_{11}^{(i)}(\alpha r) + lT_{13}^{(i)}(\beta r)\right] P_n^m(\cos\theta)\exp[i(m\phi-\omega t)]\,, \tag{42}$$

$$\begin{aligned} \sigma_{\theta\theta} = \frac{2\mu}{r^2}\Big\{ & T_{21}^{(i)}(\alpha r)P_n^m(\cos\theta) \\ & + \hat{T}_{21}^{(i)}(\alpha r)\frac{1}{\sin^2\theta}\left[\left(m^2 - n\cos^2\theta\right)P_n^m(\cos\theta) + (n+m)\cos\theta P_{n-1}^m(\cos\theta)\right] \\ & + T_{22}^{(i)}(\beta r)\frac{im}{\sin^2\theta}\left[(n-1)\cos\theta P_n^m(\cos\theta) - (n+m)P_{n-1}^m(\cos\theta)\right] + lT_{23}^{(i)}(\beta r)P_n^m(\cos\theta) \\ & + l\hat{T}_{23}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(m^2 - n\cos^2\theta\right)P_n^m(\cos\theta) + (m+n)\cos\theta P_{n-1}^m(\cos\theta)\right]\Big\} \\ & \times\exp[i(m\phi-\omega t)]\,, \end{aligned} \tag{43}$$

$$\begin{aligned} \sigma_{\phi\phi} = \frac{2\mu}{r^2}\Big\{ & T_{31}^{(i)}(\alpha r)P_n^m(\cos\theta) \\ & + \hat{T}_{31}^{(i)}(\alpha r)\frac{1}{\sin^2\theta}\left[\left(n\cos^2\theta - m^2\right)P_n^m(\cos\theta) - (n+m)\cos\theta P_{n-1}^m(\cos\theta)\right] \\ & + T_{32}^{(i)}(\beta r)\frac{im}{\sin^2\theta}\left[-(n-1)\cos\theta P_n^m(\cos\theta) + (n+m)P_{n-1}^m(\cos\theta)\right] + lT_{33}^{(i)}(\beta r)P_n^m(\cos\theta) \\ & + l\hat{T}_{33}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(n\cos^2\theta - m^2\right)P_n^m(\cos\theta) - (n+m)\cos\theta P_{n-1}^m(\cos\theta)\right]\Big\} \\ & \times\exp[i(m\phi-\omega t)]\,, \end{aligned} \tag{44}$$

$$\begin{aligned} \sigma_{r\theta} = \frac{2\mu}{r^2}\Big\{ & T_{41}^{(i)}(\alpha r)\left[n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta)\right] + T_{42}^{(i)}(\beta r)\frac{im}{\sin\theta}P_n^m(\cos\theta) \\ & + lT_{43}^{(i)}(\beta r)\left[n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta)\right]\Big\}\exp[i(m\phi-\omega t)]\,, \end{aligned} \tag{45}$$

$$\sigma_{r\phi} = \frac{2\mu}{r^2}\left\{T_{51}^{(i)}(\alpha r)\frac{im}{\sin\theta}P_n^m(\cos\theta) - T_{52}^{(i)}(\beta r)\frac{1}{\sin\theta}\left[n\cos\theta P_n^m(\cos\theta) - (n+m)P_{n-1}^m(\cos\theta)\right]\right.$$
$$\left. + lT_{53}^{(i)}(\beta r)\frac{im}{\sin\theta}P_n^m(\cos\theta)\right\}\exp[i(m\phi - \omega t)], \tag{46}$$

$$\sigma_{\theta\phi} = \frac{2\mu}{r^2}\left\{T_{61}^{(i)}(\alpha r)\frac{im}{\sin^2\theta}\left[(n-1)\cos\theta P_n^m(\cos\theta) - (n+m)P_{n-1}^m(\cos\theta)\right]\right.$$
$$+ T_{62}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(\frac{n(n-1)}{2}\sin^2\theta + n - m^2\right)P_n^m(\cos\theta) - (n+m)\cos\theta P_{n-1}^m(\cos\theta)\right]$$
$$\left. + lT_{63}^{(i)}(\beta r)\frac{im}{\sin^2\theta}\left[(n-1)\cos\theta P_n^m(\cos\theta) - (n+m)P_{n-1}^m(\cos\theta)\right]\right\}\exp[i(m\phi - \omega t)], \tag{47}$$

where

$$T_{11}^{(i)}(\alpha r) = \left(n^2 - n - \frac{1}{2}\beta^2 r^2\right)Z_n^{(i)}(\alpha r) + 2\alpha r Z_{n+1}^{(i)}(\alpha r), \tag{48}$$

$$T_{13}^{(i)}(\beta r) = n(n+1)\left[(n-1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{49}$$

$$T_{21}^{(i)}(\alpha r) = \left(-n^2 - \frac{1}{2}\beta^2 r^2 + \alpha^2 r^2\right)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{50}$$

$$T_{21}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \qquad T_{22}^{(i)}(\beta) = rZ_n^{(i)}(\beta r), \tag{51}$$

$$T_{23}^{(i)}(\beta r) = -(n^2 + n)\left[nZ_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{52}$$

$$\hat{T}_{23}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r), \tag{53}$$

$$T_{31}^{(i)}(\beta r) = \left(n - \frac{1}{2}\beta^2 r^2 + \alpha^2 r^2\right)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{54}$$

$$\hat{T}_{31}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \qquad T_{32}^{(i)}(\beta r) = rZ_n^{(i)}(\beta r), \tag{55}$$

$$T_{33}^{(i)}(\beta r) = n(n+1)Z_n^{(i)}(\beta r), \qquad \hat{T}_{33}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r), \tag{56}$$

$$T_{41}^{(i)}(\alpha r) = (n-1)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{57}$$

$$T_{42}^{(i)}(\beta r) = \frac{1}{2}r\left[(n-1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{58}$$

$$T_{43}^{(i)}(\beta r) = \left(n^2 - 1 - \frac{1}{2}\beta^2 r^2\right)Z_n^{(i)}(\beta r) + \beta r Z_{n+1}^{(i)}(\beta r), \tag{59}$$

$$T_{51}^{(i)}(\alpha r) = T_{41}^{(i)}(\alpha r) = (n-1)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{60}$$

$$T_{52}^{(i)}(\beta r) = T_{42}^{(i)}(\beta r) = \frac{1}{2}r\left[(n-1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{61}$$

$$T_{53}^{(i)}(\beta r) = T_{43}^{(i)}(\beta r) = \left(n^2 - 1 - \frac{1}{2}\beta^2 r^2\right)Z_n^{(i)}(\beta r) + \beta r Z_{n+1}^{(i)}(\beta r), \tag{62}$$

$$T_{61}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \qquad T_{62}^{(i)}(\beta r) = rZ_n^{(i)}(\beta r), \tag{63}$$

$$T_{63}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r). \tag{64}$$

$T_{k1}^{(i)}$ and $\hat{T}_{k1}^{(i)}$ correspond to the function Φ, $T_{k2}^{(i)}$ and $\hat{T}_{k2}^{(i)}$ correspond to the function Ψ while $T_{k3}^{(i)}$ and $\hat{T}_{k3}^{(i)}$ correspond to the function χ.

2.3. Modal characteristic equation

In this Section we first specialize the general solution to the elasticity equations in spherical coordinates to the axisymmetric case. Next we apply the traction-free boundary conditions to arrive at a series of modal characteristic equations for the vibrating sphere problem.

Assumption on the axisymmetric form of the vibrations (the axis of symmetry coincides with the vertical axis $\theta = 0$) implies elimination of the ϕ-component of the displacement field, $u_\phi = 0$, and all derivatives with respect to the ϕ variable, $\frac{\partial}{\partial \phi} = 0$. Consequently, $q = 0$ in Eq. (26), solution F_3 reduces to a constant function and $m = 0$ in Legendre's equation (23), i.e. the associated Legendre functions $P_n^m(\mu)$ reduce just to the Legendre polynomials $P_n(\mu)$.

Summarizing, the formulas for the Helmholtz potentials in the axisymmetric case reduce to

$$\Phi = Z_n^{(i)}(\alpha r) P_n(\cos\theta) \exp(-i\omega t)\,, \tag{65}$$

$$\Psi = Z_n^{(i)}(\beta r) \exp(-i\omega t)\,, \tag{66}$$

$$\chi = Z_n^{(i)}(\beta r) P_n(\cos\theta) \exp(-i\omega t)\,, \tag{67}$$

with

$$Z_n^{(1)} \equiv j_n(kr) \equiv \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kr)\,, \tag{68}$$

$$Z_n^{(2)} \equiv y_n(kr) \equiv \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} Y_{n+\frac{1}{2}}(kr)\,, \tag{69}$$

$$\alpha \equiv \frac{\omega}{c_1}, \qquad \beta \equiv \frac{\omega}{c_2}\,, \tag{70}$$

where P_n are the Legendre polynomials and $J_{n+\frac{1}{2}}$ are the Bessel functions of order $n + \frac{1}{2}$. The corresponding displacement field takes the simplified form

$$u_r = \frac{\partial \Phi}{\partial r} + l\left[\frac{\partial^2 (r\chi)}{\partial r^2} - r\nabla^2\chi\right], \tag{71}$$

$$u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{l}{r}\frac{\partial^2 (r\chi)}{\partial\theta\partial r}\,, \tag{72}$$

$$u_\phi = 0\,, \tag{73}$$

and the corresponding stress field looks as follows:

$$\sigma_{rr} = \lambda\nabla^2\Phi + 2\mu\frac{\partial^2\Phi}{\partial r^2} + 2\mu l\frac{\partial}{\partial r}\left[\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right], \tag{74}$$

$$\sigma_{\theta\theta} = \lambda\nabla^2\Phi + 2\mu\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) + 2\mu l\left[\frac{1}{r^2}\frac{\partial^3(r\chi)}{\partial\theta^2\partial r} + \frac{1}{r}\left(\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right)\right], \tag{75}$$

$$\sigma_{\phi\phi} = \lambda\nabla^2\Phi + 2\mu\left[\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\cot\theta\frac{\partial\Phi}{\partial\theta}\right] + 2\mu l\left[\frac{1}{r}\left(\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right) + \frac{\cot\theta}{r^2}\frac{\partial^2(r\chi)}{\partial\theta\partial r}\right], \tag{76}$$

$$\sigma_{r\theta} = \frac{2\mu}{r}\left(\frac{\partial^2\Phi}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) + \frac{\mu l}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial^2(r\chi)}{\partial r^2} - r\nabla^2\chi\right) - \frac{1}{r}\frac{\partial^2(r\chi)}{\partial\theta\partial r} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2(r\chi)}{\partial\theta\partial r}\right)\right], \tag{77}$$

$$\sigma_{r\phi} = 0\,, \tag{78}$$

$$\sigma_{\theta\phi} = 0\,. \tag{79}$$

Substituting Φ, Ψ and χ in (65), (66) and (67) into the stress field above, we arrive at the final formulas for stresses in terms of the Bessel functions and Legendre polynomials,

$$\sigma_{rr} = \frac{2\mu}{r^2}\left[T_{11}^{(i)}(\alpha r) + lT_{13}^{(i)}(\beta r)\right] P_n(\cos\theta)\exp(-i\omega t)\,, \tag{80}$$

$$\sigma_{\theta\theta} = \frac{2\mu}{r^2}\left\{T_{21}^{(i)}(\alpha r)P_n(\cos\theta) + \hat{T}_{21}^{(i)}(\alpha r)\frac{1}{\sin^2\theta}[-n\cos^2\theta P_n(\cos\theta) + n\cos\theta P_{n-1}(\cos\theta)]\right.$$
$$\left. + lT_{23}^{(i)}(\beta r)P_n(\cos\theta) + l\hat{T}_{23}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(-n\cos^2\theta\right)P_n(\cos\theta) + n\cos\theta P_{n-1}(\cos\theta)\right]\right\}$$
$$\times\exp(-iwt), \tag{81}$$

$$\sigma_{\phi\phi} = \frac{2\mu}{r^2}\left\{T_{31}^{(i)}(\alpha r)P_n(\cos\theta) + \hat{T}_{31}^{(i)}(\alpha r)\frac{1}{\sin^2\theta}[n\cos^2\theta P_n(\cos\theta) - n\cos\theta P_{n-1}(\cos\theta)]\right.$$
$$\left. + lT_{33}^{(i)}(\beta r)P_n(\cos\theta) + l\hat{T}_{33}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(n\cos^2\theta\right)P_n(\cos\theta) - n\cos\theta P_{n-1}(\cos\theta)\right]\right\}$$
$$\times\exp(-iwt), \tag{82}$$

$$\sigma_{r\theta} = \frac{2\mu}{r^2}\left\{T_{41}^{(i)}(\alpha r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\right.$$
$$\left. + lT_{43}^{(i)}(\beta r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\right\}\exp(-iwt), \tag{83}$$

$$\sigma_{r\phi} = \sigma_{\theta\phi} = 0, \tag{84}$$

where:

$$T_{11}^{(i)}(\alpha r) = \left(n^2 - n - \frac{1}{2}\beta^2 r^2\right)Z_n^{(i)}(\alpha r) + 2\alpha r Z_{n+1}^{(i)}(\alpha r), \tag{85}$$

$$T_{13}^{(i)}(\beta r) = n(n+1)\left[(n-1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{86}$$

$$T_{21}^{(i)}(\alpha r) = \left(-n^2 - \frac{1}{2}\beta^2 r^2 + \alpha^2 r^2\right)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{87}$$

$$\hat{T}_{21}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \tag{88}$$

$$T_{23}^{(i)}(\beta r) = -(n^2+n)\left[nZ_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{89}$$

$$\hat{T}_{23}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r), \tag{90}$$

$$T_{31}^{(i)}(\beta r) = \left(n - \frac{1}{2}\beta^2 r^2 + \alpha^2 r^2\right)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{91}$$

$$\hat{T}_{31}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \tag{92}$$

$$T_{33}^{(i)}(\beta r) = n(n+1)Z_n^{(i)}(\beta r), \tag{93}$$

$$\hat{T}_{33}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r), \tag{94}$$

$$T_{41}^{(i)}(\alpha r) = (n-1)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{95}$$

$$T_{43}^{(i)}(\beta r) = \left(n^2 - 1 - \frac{1}{2}\beta^2 r^2\right)Z_n^{(i)}(\beta r) + \beta r Z_{n+1}^{(i)}(\beta r). \tag{96}$$

Traction boundary conditions

The boundary conditions on the inner surface and on the outer surface of the hollow sphere are

$$\sigma_{rr} = \sigma_{r\phi} = \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_i, \tag{97}$$

$$\sigma_{rr} = \sigma_{r\phi} = \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_o, \tag{98}$$

where r_i is the inner radius and r_o is the outer radius. Boundary conditions $\sigma_{r\phi} = 0$ on both the inner and outer surfaces are automatically satisfied. The remaining four boundary conditions contribute with the following four equations in terms of unknown coefficients $A = A_n$, $B = B_n$, $C = C_n$, $D = D_n$ for every $n = 0, 1, 2, \ldots$

$$\sigma_{rr}|_{r=r_i} = \frac{2\mu}{r_i^2}\left[AT_{11}^{(1)}(\alpha r_i)+BlT_{13}^{(1)}(\beta r_i)+CT_{11}^{(2)}(\alpha r_i)+DlT_{13}^{(2)}(\beta r_i)\right]P_n(\cos\theta)\exp(-iwt) = 0\,, \tag{99}$$

$$\sigma_{rr}|_{r=r_o} = \frac{2\mu}{r_o^2}\left[AT_{11}^{(1)}(\alpha r_o)+BlT_{13}^{(1)}(\beta r_o)+CT_{11}^{(2)}(\alpha r_o)+DlT_{13}^{(2)}(\beta r_o)\right]P_n(\cos\theta)\exp(-iwt) = 0\,, \tag{100}$$

$$\begin{aligned}\sigma_{r\theta}|_{r=r_i} = \frac{2\mu}{r_i^2}\Big\{&AT_{41}^{(1)}(\alpha r_i)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\\ &+ BlT_{43}^{(1)}(\beta r_i)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\\ &+ CT_{41}^{(2)}(\alpha r_i)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{(n-1)}(\cos\theta)\right]\\ &+ DlT_{43}^{(2)}(\beta r_i)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\Big\}e^{-iwt} = 0\,,\end{aligned} \tag{101}$$

$$\begin{aligned}\sigma_{r\theta}|_{r=r_o} = \frac{2\mu}{r_o^2}\Big\{&AT_{41}^{(1)}(\alpha r_o)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\\ &+ BlT_{43}^{(1)}(\beta r_o)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\\ &+ CT_{41}^{(2)}(\alpha r_o)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{(n-1)}(\cos\theta)\right]\\ &+ DlT_{43}^{(2)}(\beta r_o)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\Big\}e^{-iwt} = 0\,.\end{aligned} \tag{102}$$

Modal characteristic equation

Requesting a non-trivial solution to (99)–(102), we arrive at the modal characteristic equation in the form

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_i) & T_{13}^{(1)}(\beta r_i) & T_{11}^{(2)}(\alpha r_i) & T_{13}^{(2)}(\beta r_i) \\ T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ T_{41}^{(1)}(\alpha r_i) & T_{43}^{(1)}(\beta r_i) & T_{41}^{(2)}(\alpha r_i) & T_{43}^{(2)}(\beta r_i) \\ T_{41}^{(1)}(\alpha r_o) & T_{43}^{(1)}(\beta r_o) & T_{41}^{(2)}(\alpha r_o) & T_{43}^{(2)}(\beta r_o) \end{vmatrix} = 0 \quad \text{for } n > 0\,, \tag{103}$$

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_i) & T_{11}^{(2)}(\alpha r_i) \\ T_{11}^{(1)}(\alpha r_o) & T_{11}^{(2)}(\alpha r_o) \end{vmatrix} = 0 \quad \text{for } n = 0\,, \tag{104}$$

where

$$\alpha = \frac{\omega}{c_1}, \quad \beta = \frac{\omega}{c_2}, \quad c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}\,.$$

Note that for $n = 0$, Equations (101) and (102) are automatically satisfied, so that the determinant has the form (104).

2.4. A stable algorithm for the evaluation of Bessel functions

Many algorithms have been proposed to evaluate Bessel functions of fractional order. Here we adopt the Steed's method and Temme's Series. The Steed method described in [5] consists in calculating J_ν, J'_ν, Y_ν and Y'_ν using the following three relations:

- Wronskian Relation

$$W \equiv J_\nu Y'_\nu - Y_\nu J'_\nu = \frac{2}{\pi x} \tag{105}$$

- First continued fraction (CF1)

$$f_\nu \equiv \frac{J'_\nu}{J_\nu} = \frac{\nu}{x} - \frac{J_{\nu+1}}{J_\nu} = \frac{\nu}{x} - \frac{1}{2(\nu+1)/x-}\,\frac{1}{2(\nu+1)/x-}\cdots \tag{106}$$

The rate of convergence for CF1 is determined by the position of the turning point $x_{tp} = \sqrt{\nu(\nu+1)} \approx \nu$. If $x <\approx x_{tp}$, the convergence of CF1 is very rapid. If $x >\approx x_{tp}$, each iteration of CF1 effectively increases ν by one until $x <\approx x_{tp}$.

- Second continued fraction (CF2)

$$p + iq \equiv \frac{J'_\nu + iY'_\nu}{J_\nu + iY_\nu} = -\frac{1}{2x} + i + \frac{i}{x}\,\frac{\left(\frac{1}{2}\right)^2 - \nu^2}{2(x+i)+}\,\frac{\left(\frac{3}{2}\right)^2 - \nu^2}{2(x+2i)+}\cdots \tag{107}$$

If $x >\approx x_{tp}$ then Eq. (107) converges rapidly.

For x not small, we can ensure that $x >\approx x_{tp}$ by stable downward recurrence J_ν and J'_ν to a value $\nu = \mu <\approx x$. The initial values for the recurrence are

$$J_\nu = \text{arbitrary}\,, \tag{108}$$

$$J'_{\nu-1} = f_\nu J_\nu\,. \tag{109}$$

The downward recurrence relations are

$$J_{\nu-1} = \frac{\nu}{x} J_\nu + J'_\nu\,, \tag{110}$$

$$J'_{\nu-1} = \frac{\nu-1}{x} J_{\nu-1} + J_\nu\,. \tag{111}$$

Since CF2 is evaluated at $\nu = \mu$, from Eqs. (105), (106) and (107) we can solve the equation for four unknowns, J_μ, J'_μ, Y_μ and Y'_μ

$$J_\mu = \pm\left(\frac{W}{q + \gamma(p - f_\mu)}\right)^{\frac{1}{2}}, \tag{112}$$

$$J'_\mu = f_\mu J_\mu\,, \tag{113}$$

$$Y_\mu = \gamma J_\mu\,, \tag{114}$$

$$Y'_\mu = Y_\mu\left(p + \frac{q}{\gamma}\right), \tag{115}$$

where the sign of J_μ is the same as that of the initial J_ν in Eq. (108) and

$$\gamma = \frac{p - f_\mu}{q}\,. \tag{116}$$

Once four functions have been determined at $\nu = \mu$, the quantities of Y_ν and Y'_ν can be evaluated by the stable upwards recurrence formula. By scaling the value in Eq. (108) according to the ratio of J_μ in (112) to the value found after recurrence calculation in (110), we can obtain J_ν and J'_ν.

For the case of small x, the convergence of the second continued fraction will fail. However, Temme's series can be applied to give the good estimate of Y_ν and $Y_{\nu+1}$ as

$$Y_\nu = -\sum_{k=0}^{\infty} c_k g_k \,, \qquad Y_{\nu+1} = -\frac{2}{x}\sum_{k=0}^{\infty} c_k h_k \,, \tag{117}$$

where

$$c_k = \frac{(-x^2/4)^k}{k!}, \tag{118}$$

$$g_k = f_k + \frac{2}{\nu}\sin^2\left(\frac{\nu x}{2}\right) q_k \,, \tag{119}$$

$$h_k = -k g_k + p_k \,, \tag{120}$$

$$p_k = \frac{p_{k-1}}{k-\nu}\,, \tag{121}$$

$$q_k = \frac{q_{k-1}}{k+\nu}\,, \tag{122}$$

$$f_k = \frac{k f_{k-1} + p_{k-1} + q_{k-1}}{k^2 - \nu^2}\,, \tag{123}$$

$$p_0 = \frac{1}{\pi}\left(\frac{x}{2}\right)^{-\nu} \Gamma(1+\nu)\,, \tag{124}$$

$$q_0 = \frac{1}{\pi}\left(\frac{x}{2}\right)^{\nu} \Gamma(1-\nu)\,, \tag{125}$$

$$f_0 = \frac{2}{\pi}\frac{\nu\pi}{\sin\nu\pi}\left[\cosh\sigma\Gamma_1(\nu) + \frac{\sinh\sigma}{\sigma}\ln\left(\frac{2}{x}\right)\Gamma_2(\nu)\right], \tag{126}$$

$$\sigma = \nu \ln\frac{2}{x}\,, \tag{127}$$

$$\Gamma_1(\nu) = \frac{1}{2\nu}\left[\frac{1}{\Gamma(1-\nu)} - \frac{1}{\Gamma(1+\nu)}\right], \tag{128}$$

$$\Gamma_2(\nu) = \frac{1}{2}\left[\frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(1+\nu)}\right]. \tag{129}$$

For more detail of these methods see [5].

3. Kirchhoff-Love shell theory solution

Following [3] we review the classical Kirchhoff-Love theory for thin shells and derive the shell equations in spherical coordinates under the simplifying assumption of axisymmetric vibrations. As in the case of 3-D continuum, we arrive finally at a series of characteristic modal equations.

3.1. A review of the shell theory

In order to derive the equations of motion of thin elastic shells, Love introduced the following four assumptions:

- $h/a \ll 1$, i.e., thickness h over midsurface radius a is very small,
- u_r/h, u_θ/h and $u_\phi/h \ll 1$, i.e. the displacement is small compared with thickness,

- σ_{rr} is negligible,
- Fibers in the radial direction remain undeformed during the motion.

Based on these assumptions, if we consider the axisymmetric vibrations only, we can express the components of the displacement vector in terms of the displacements of middle surface

$$u_r = U_r \,, \tag{130}$$

$$u_\theta = \left(1+\frac{x}{a}\right)U_\theta - \frac{x}{a}\frac{\partial U_r}{\partial \theta}\,, \tag{131}$$

where $x = r - a$ and U_r as well as U_θ are functions of θ only.

The kinetic energy can be expressed as follows:

$$T = \frac{1}{2}\rho_s \int_0^{2\pi}\int_0^{\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\dot{u_r}^2 + \dot{u_\theta}^2\right)(a+x)^2 \sin\theta \,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{132}$$

After neglecting x in comparison to midsurface radius a, we can obtain the total kinetic energy in terms of u_r and u_θ.

$$T = \frac{1}{2}\rho_s \int_0^{2\pi}\int_0^{\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\dot{u_r}^2 + \dot{u_\theta}^2\right)a^2 \sin\theta \,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{133}$$

Neglecting the effects of rotatory inertia and plugging (130) and (131) into (133), we can simplify the total kinetic energy to the following form

$$T = \pi\rho_s h a^2 \int_0^{\pi}\left(\dot{U_r}^2 + \dot{U_\theta}^2\right)\sin\theta\,\mathrm{d}\theta\,. \tag{134}$$

Non-vanishing components of strain in spherical coordinates can be expressed in terms of u_r and u_θ as

$$\varepsilon_{\theta\theta} = \frac{1}{a+x}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right), \tag{135}$$

$$\varepsilon_{\phi\phi} = \frac{1}{a+x}(\cot\theta u_\theta + u_r)\,. \tag{136}$$

If we substitute (130) and (131) into (135) and (136), then we get

$$\varepsilon_{\theta\theta} = \frac{1}{a+x}\left(\frac{\partial U_\theta}{\partial\theta} + U_r\right) + \frac{x}{a(a+x)}\left(\frac{\partial U_\theta}{\partial\theta} - \frac{\partial^2 U_r}{\partial\theta^2}\right), \tag{137}$$

$$\varepsilon_{\phi\phi} = \frac{1}{a+x}(\cot\theta U_\theta + U_r) + \frac{x}{a(a+x)}\cot\theta\left(U_\theta - \frac{\partial U_r}{\partial\theta}\right). \tag{138}$$

The non-vanishing components of stress in terms of $\varepsilon_{\theta\theta}$ and $\varepsilon_{\phi\phi}$ are

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\varepsilon_{\phi\phi})\,, \tag{139}$$

$$\sigma_{\phi\phi} = \frac{E}{1-\nu^2}(\varepsilon_{\phi\phi} + \nu\varepsilon_{\theta\theta})\,. \tag{140}$$

Using Eqs. (137)–(140), we get now the strain energy of the shell as

$$\begin{aligned}
V &= \frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{\phi\phi}\varepsilon_{\phi\phi})(a+x)^2\sin\theta\,\mathrm{d}x\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{\pi E h}{1-\nu^2}\int_0^{\pi}\left\{\left(\frac{\partial U_\theta}{\partial\theta} + U_r\right)^2 + (\cot\theta U_\theta + U_r)^2 + 2\nu\left(\frac{\partial U_\theta}{\partial\theta} + U_r\right)(\cot\theta U_\theta + U_r)\right\}\sin\theta\,\mathrm{d}\theta \\
&\quad + \frac{\pi E h}{1-\nu^2}\beta^2\int_0^{\pi}\left\{\left(\frac{\partial U_\theta}{\partial\theta} - \frac{\partial^2 U_r}{\partial\theta^2}\right)^2 + \cot^2\theta\left(U_\theta - \frac{\partial U_r}{\partial\theta}\right)^2\right. \\
&\qquad\qquad \left. + 2\nu\cot\theta\left(\frac{\partial U_\theta}{\partial\theta} - \frac{\partial^2 U_r}{\partial\theta^2}\right)\left(U_\theta - \frac{\partial U_r}{\partial\theta}\right)\right\}\sin\theta\,\mathrm{d}\theta
\end{aligned} \tag{141}$$

where

$$\beta^2 = \frac{h^2}{12a^2}$$

Applying the Hamilton principle, we get the final equations of motion

$$\begin{aligned}
&\left(1+\beta^2\right)\left[\frac{\partial^2 U_\theta}{\partial\theta^2} + \cot\theta\frac{\partial U_\theta}{\partial\theta} - \left(\nu + \cot^2\theta\right)U_\theta\right] \\
&\qquad -\beta^2\frac{\partial^3 U_r}{\partial\theta^3} - \beta^2\cot\theta\frac{\partial^2 U_r}{\partial\theta^2} + \left[(1+\nu) + \beta^2\left(\nu+\cot^2\theta\right)\right]\frac{\partial U_r}{\partial\theta} - \frac{a^2\ddot{U}_\theta}{c_p^2} = 0\,, \qquad (142)
\end{aligned}$$

$$\begin{aligned}
&\beta^2\frac{\partial^3 U_\theta}{\partial\theta^3} + 2\beta^2\cot\theta\frac{\partial^2 U_\theta}{\partial\theta^2} - \left[(1+\nu)\left(1+\beta^2\right) + \beta^2\cot^2\theta\right]\frac{\partial U_\theta}{\partial\theta} \\
&\quad + \cot\theta\left[\left(2-\nu+\cot^2\theta\right)\beta^2 - (1+\nu)\right]U_\theta - \beta^2\frac{\partial^4 U_r}{\partial\theta^4} - 2\beta^2\cot\theta\frac{\partial^3 U_r}{\partial\theta^3} \\
&\quad + \beta^2\left(1+\nu+\cot^2\theta\right)\frac{\partial^2 U_r}{\partial\theta^2} - \beta^2\cot\theta\left(2-\nu+\cot^2\theta\right)\frac{\partial U_r}{\partial\theta} - 2(1+\nu)U_r - \frac{a^2\ddot{U}_r}{c_p^2} = 0\,, \qquad (143)
\end{aligned}$$

or, in a more compact form,

$$L_{\theta\theta}U_\theta + L_{\theta r}U_r + \Omega^2 U_\theta = 0\,, \qquad (144)$$

$$L_{r\theta}U_\theta + L_{rr}U_r + \Omega^2 U_r = 0\,, \qquad (145)$$

where

$$L_{\theta\theta} = \left(1+\beta^2\right)\left\{\left(1-\eta^2\right)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(1-\eta^2\right)^{\frac{1}{2}} + (1-\nu)\right\}\,, \qquad (146)$$

$$L_{\theta r} = \left(1-\eta^2\right)^{\frac{1}{2}}\left\{\left[\beta^2(1-\nu) - (1+\nu)\right]\frac{\mathrm{d}}{\mathrm{d}\eta} + \beta^2\frac{\mathrm{d}}{\mathrm{d}\eta}\nabla_\eta^2\right\}\,, \qquad (147)$$

$$L_{r\theta} = -\left\{\left[\beta^2(1-\nu) - (1+\nu)\right]\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1-\eta^2\right)^{\frac{1}{2}} + \beta^2\nabla_\eta^2\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1-\eta^2\right)^{\frac{1}{2}}\right\}\,, \qquad (148)$$

$$L_{rr} = -\beta^2\nabla_\eta^4 - \beta^2(1-\nu)\nabla_\eta^2 - 2(1+\nu) \qquad (149)$$

and

$$\nabla_\eta^2 = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(1-\eta^2\right)\frac{\mathrm{d}}{\mathrm{d}\eta}\,. \qquad (150)$$

The following notation has been used

- a – the radius of the middle surface of the shell,
- E, ν – the Young modulus and Poisson ratio,
- h – the shell thickness,
- $\eta = \cos\theta$,
- Ω – the dimensionless frequency of the shell, $\Omega = \frac{wa}{c_p} = (\frac{c}{c_p})ka$,
- c – wave velocity,
- c_p – the low frequency phase velocity of compressional waves in an elastic plate,
- ω – the frequency,
- k – the wave number.

3.2. Free vibration problem

The displacement field admits the spectral representation in terms of the Legendre polynomials:

$$U_r(\eta) = \sum_{n=0}^{\infty} U_{rn} P_n(\eta) \exp(-iwt), \tag{151}$$

$$U_\theta(\eta) = \sum_{n=0}^{\infty} U_{\theta n} \left(1-\eta^2\right)^{\frac{1}{2}} \frac{\mathrm{d}P_n}{\mathrm{d}\eta} \exp(-iwt), \tag{152}$$

By substituting (151) and (152) into (142) and (143), we can obtain

$$\left[\Omega^2 - \left(1+\beta^2\right)(\nu+\lambda_n-1)\right] U_{\theta n} - \left[\beta^2(\nu+\lambda_n-1)+(1+\nu)\right] U_{rn} = 0, \tag{153}$$

$$-\lambda_n \left[\beta^2(\nu+\lambda_n-1)+(1+\nu)\right] U_{\theta n} + \left[\Omega^2 - 2(1+\nu) - \beta^2\lambda_n(\nu+\lambda_n-1)\right] U_{rn} = 0. \tag{154}$$

For a non-trivial solution of Eqs. (153) and (154), the characteristic equation for natural frequencies Ω must be satisfied

$$\Omega^4 - \left[1+3\nu+\lambda_n - \beta^2\left(1-\nu-\lambda_n^2-\nu\lambda_n\right)\right]\Omega^2 + (\lambda_n-2)\left(1-\nu^2\right) + \beta^2\left[\lambda_n^3 - 4\lambda_n^2 + \lambda_n\left(5-\nu^2\right) - 2\left(1-\nu^2\right)\right] = 0 \tag{155}$$

where $\lambda_n = n(n+1)$, or, equivalently

$$\left[\Omega^2 - \left(\Omega_n^{(1)}\right)^2\right]\left[\Omega^2 - \left(\Omega_n^{(2)}\right)^2\right] = 0. \tag{156}$$

The higher natural frequencies are denoted by $\Omega_n^{(2)}$ and the lower natural frequencies are denoted by $\Omega_n^{(1)}$. We call the collection of $\Omega_n^{(1)}$ the first branch and that of $\Omega_n^{(2)}$ the second branch.

4. NUMERICAL EXPERIMENTS AND CONCLUSIONS

We conclude our investigations by presenting a series of numerical experiments aimed at comparing the natural frequencies of the vibrating, elastic sphere using both the full 3-D elasticity theory and the Kirchhoff-Love shell theory approximations. All comparisons are done in terms of the nondimensional natural frequency Ω' (see the definition below).

We begin by restating the modal characteristic equation of the shell theory in the form

$$\Omega^4 - \left[1+3\nu+\lambda_n - \beta^2\left(1-\nu-\lambda_n^2-\nu\lambda_n\right)\right]\Omega^2 + (\lambda_n-2)\left(1-\nu^2\right) + \beta^2\left[\lambda_n^3 - 4\lambda_n^2 + \lambda_n\left(5-\nu^2\right) - 2\left(1-\nu^2\right)\right] = 0 \tag{157}$$

or, equivalently,

$$\left[\Omega^2 - \left(\Omega_n^{(1)}\right)^2\right]\left[\Omega^2 - \left(\Omega_n^{(2)}\right)^2\right] = 0, \tag{158}$$

where

$$\lambda_n = n(n+1), \qquad \beta = \frac{1}{\sqrt{12}}\frac{h}{a}, \qquad \Omega = \frac{\omega a}{c_p}, \qquad c_p = \left[\frac{E}{(1-\nu^2)\rho}\right]^{\frac{1}{2}},$$

and the modal characteristic equation of the 3-D theory

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_i) & T_{13}^{(1)}(\beta r_i) & T_{11}^{(2)}(\alpha r_i) & T_{13}^{(2)}(\beta r_i) \\ T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ T_{41}^{(1)}(\alpha r_i) & T_{43}^{(1)}(\beta r_i) & T_{41}^{(2)}(\alpha r_i) & T_{43}^{(2)}(\beta r_i) \\ T_{41}^{(1)}(\alpha r_o) & T_{43}^{(1)}(\beta r_o) & T_{41}^{(2)}(\alpha r_o) & T_{43}^{(2)}(\beta r_o) \end{vmatrix} = 0 \quad \text{for } n > 0, \tag{159}$$

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_i) & T_{11}^{(2)}(\alpha r_i) \\ T_{11}^{(1)}(\alpha r_o) & T_{11}^{(2)}(\alpha r_o) \end{vmatrix} = 0 \quad \text{for } n = 0\,. \tag{160}$$

Six cases, corresponding to different ratios of thickness h and middle surface radius a, $h/a = 0.01$, $h/a = 0.025$, $h/a = 0.05$, $h/a = 0.1$, $h/a = 0.25$ and $h/a = 0.5$ were investigated, with $c_1 = 5760\,\text{m/sec}$, $a = 1\,\text{m}$ and $\nu = 0.29$. The results of calculations are compared in terms of the nondimensional frequency

$$\Omega' = \frac{\omega}{c_1} a\,. \tag{161}$$

Both characteristic equations were solved by using the standard bisection method. Bessel functions were evaluated using the algorithm described in Section 2.

Figures 1–6 present the comparisons of the natural frequencies obtained using both theories for the different ratios h/a. The difference becomes visible starting with $h/a = 0.05$ with the first branch of eigenfrequencies being more sensitive to the choice of equations.

The sensitivity of both branches with respect to ratio h/a is displayed in Fig. 7 for the shell theory and in Fig. 8 for the 3-D theory. The qualitative difference in the behavior of the second branch for both theories can be observed. While, for the shell theory, the second branch moves up with h/a decreasing, the same branch for the 3-D results is moving down getting closer to the first branch.

Finally, in Fig. 9 we indicate the qualitative difference between the two characteristic equations. While Equation (157), for $n \geq 1$, has two double eigenfrequencies only, Equation (159) has infinitely many solutions. This corresponds to the presence of multiple branches, with only the first two branches reproduced by the shell theory. Figure 10 presents the variation of determinant of (159) for $h/a = 0.01$ indicating the existence of the higher natural frequencies corresponding to branches of higher order.

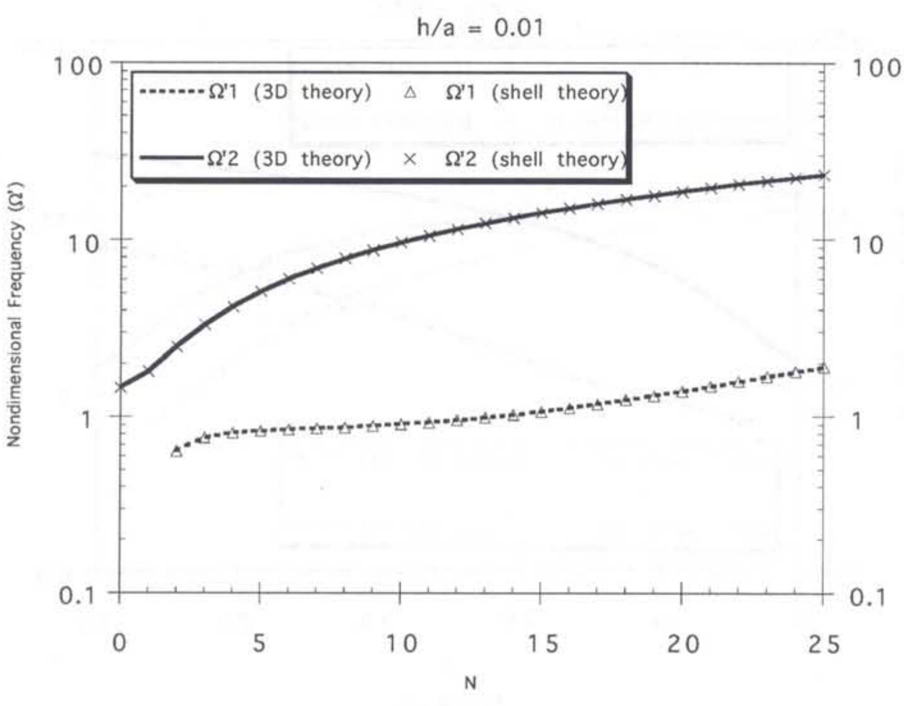

Fig. 1

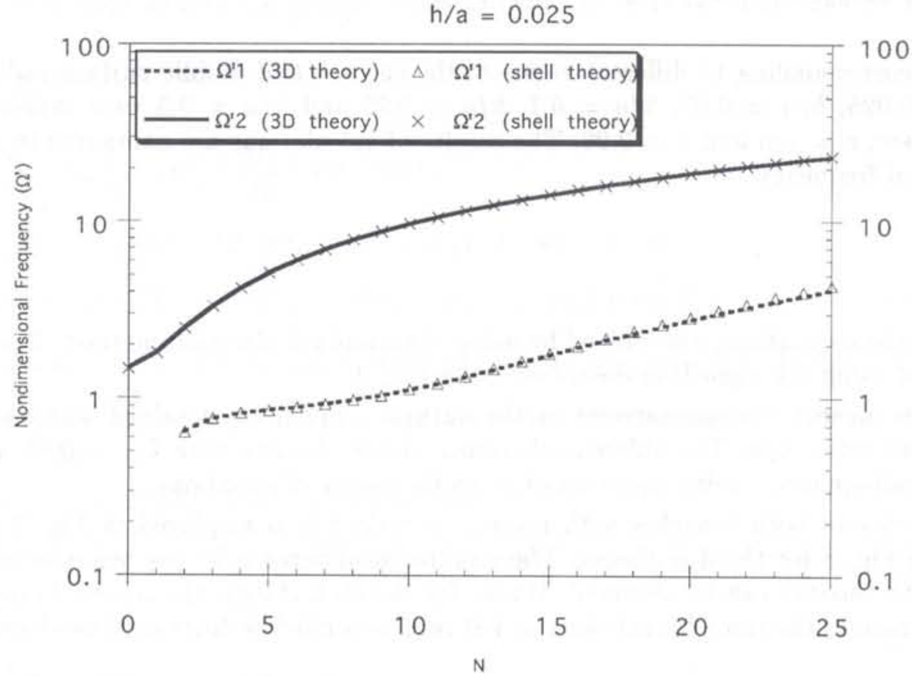
h/a = 0.025
Ω'1 (3D theory)
Ω'1 (shell theory)
Ω'2 (3D theory)
Ω'2 (shell theory)
Nondimensional Frequency (Ω')
100
10
1
0.1
0
5
10
15
20
25
N

Fig. 2

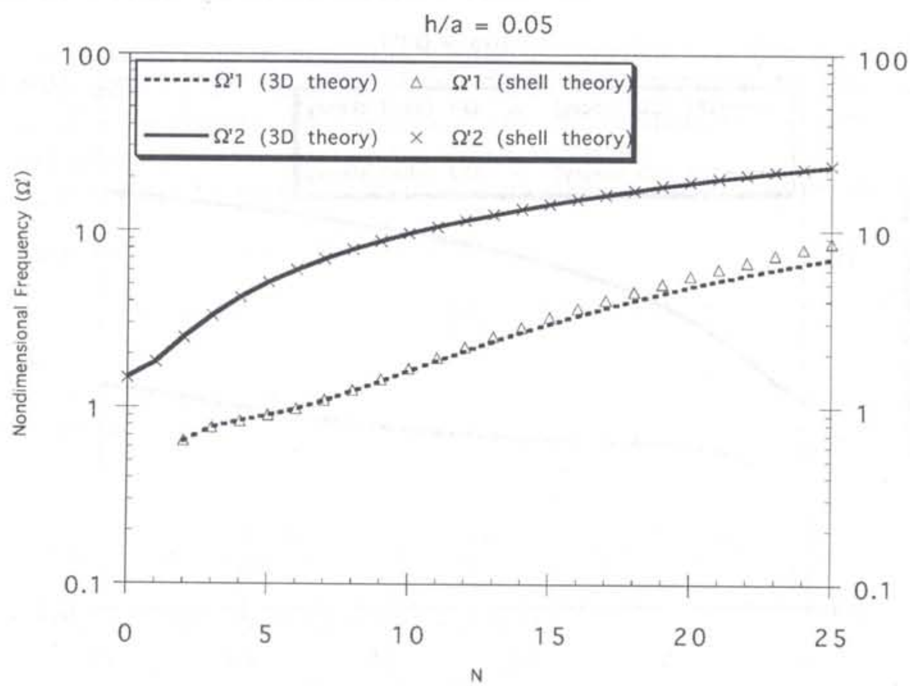
h/a = 0.05
Ω'1 (3D theory)
Ω'1 (shell theory)
Ω'2 (3D theory)
Ω'2 (shell theory)
Nondimensional Frequency (Ω')
100
10
1
0.1
0
5
10
15
20
25
N

Fig. 3

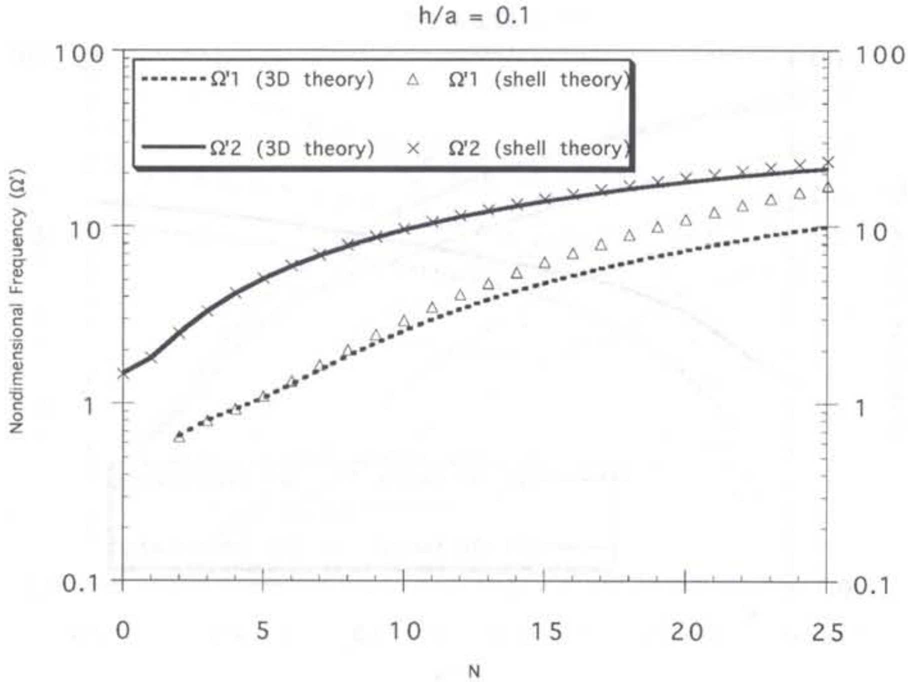
h/a = 0.1
Ω'1 (3D theory)
Ω'1 (shell theory)
Ω'2 (3D theory)
Ω'2 (shell theory)
Nondimensional Frequency (Ω')
100
10
1
0.1
0
5
10
15
20
25
N

Fig. 4

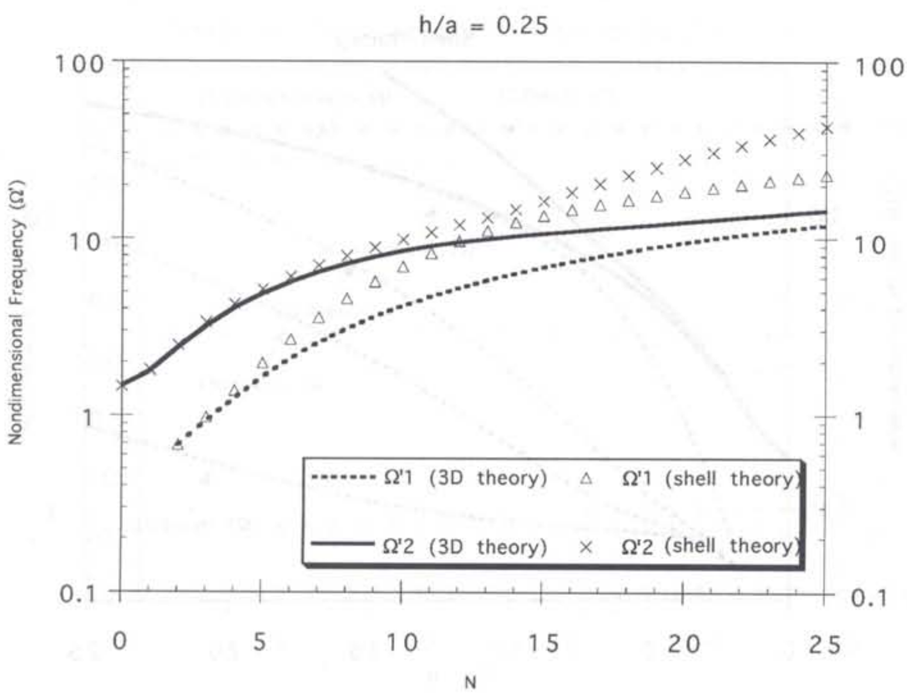
h/a = 0.25
Ω'1 (3D theory)
Ω'1 (shell theory)
Ω'2 (3D theory)
Ω'2 (shell theory)
Nondimensional Frequency (Ω')
100
10
1
0.1
0
5
10
15
20
25
N

Fig. 5

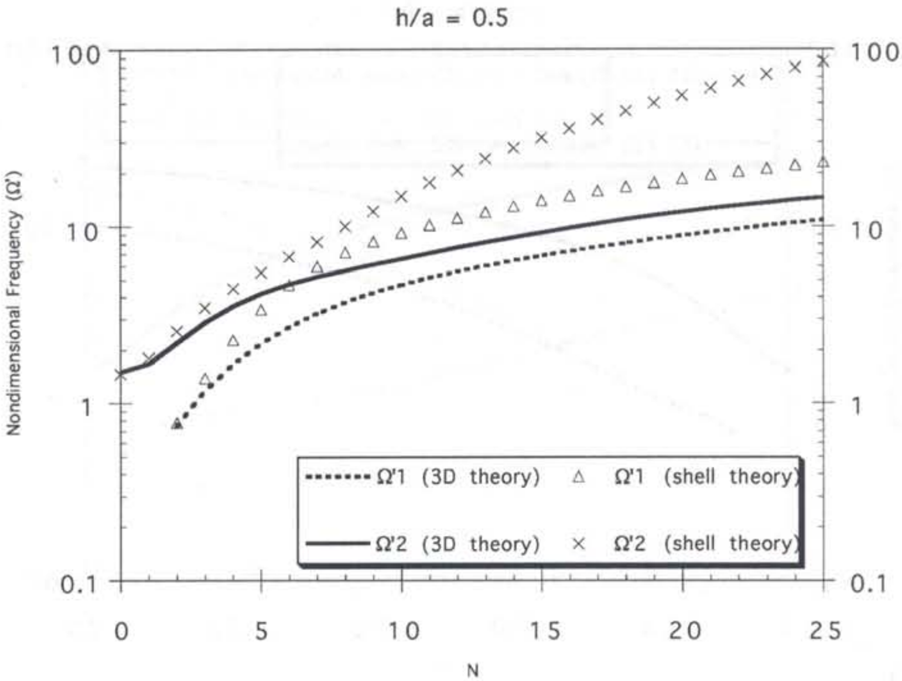
h/a = 0.5
100
10
1
0.1
Nondimensional Frequency (Ω')
Ω'1 (3D theory)
Ω'1 (shell theory)
Ω'2 (3D theory)
Ω'2 (shell theory)
0
5
10
15
20
25
N

Fig. 6

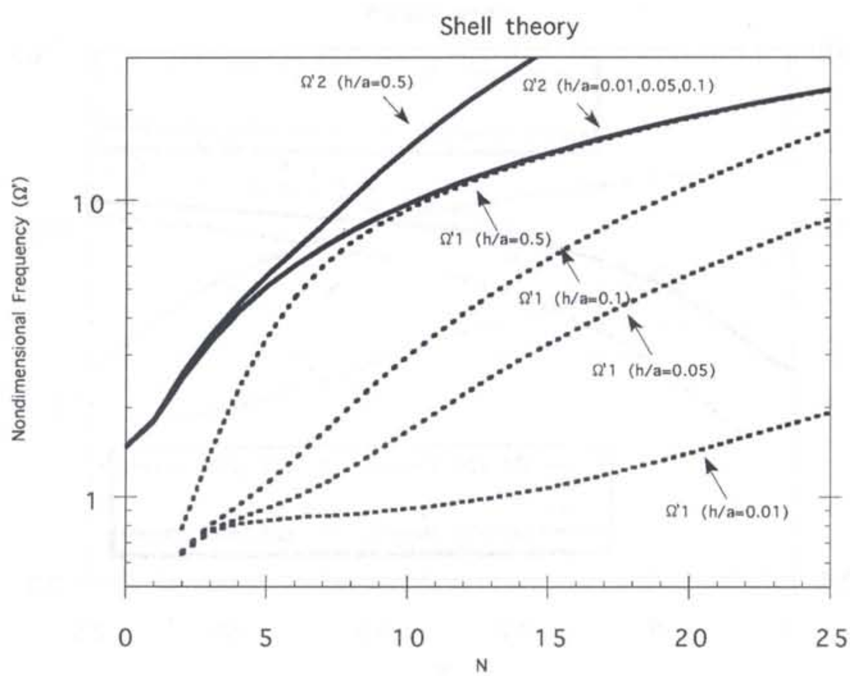
Shell theory
Ω'2 (h/a=0.5)
Ω'2 (h/a=0.01,0.05,0.1)
Ω'1 (h/a=0.5)
Ω'1 (h/a=0.1)
Ω'1 (h/a=0.05)
Ω'1 (h/a=0.01)
10
1
Nondimensional Frequency (Ω')
0
5
10
15
20
25
N

Fig. 7

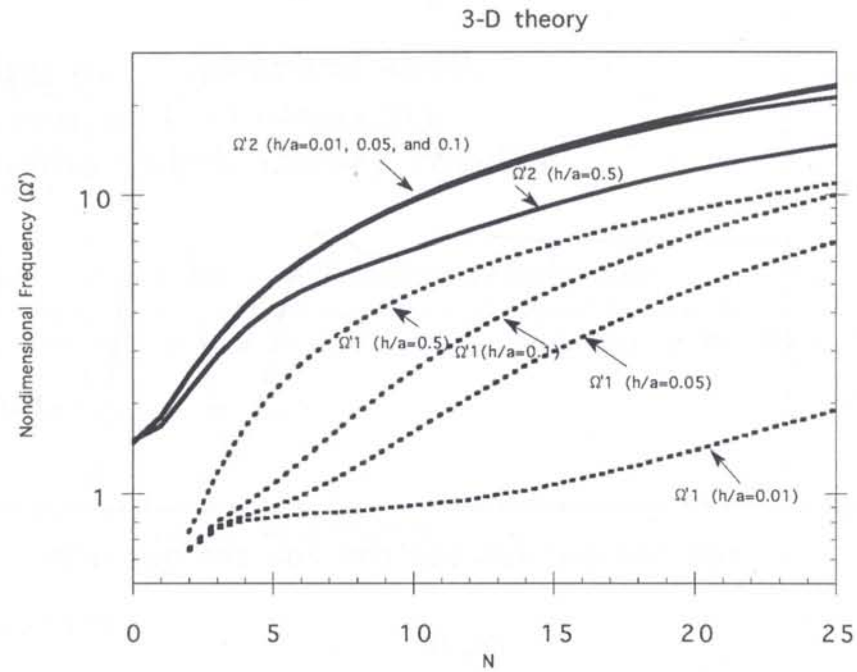
3-D theory
Ω'2 (h/a=0.01, 0.05, and 0.1)
Ω'2 (h/a=0.5)
Ω'1 (h/a=0.5)
Ω'1 (h/a=0.1)
Ω'1 (h/a=0.05)
Ω'1 (h/a=0.01)
Nondimensional Frequency (Ω')
N

Fig. 8

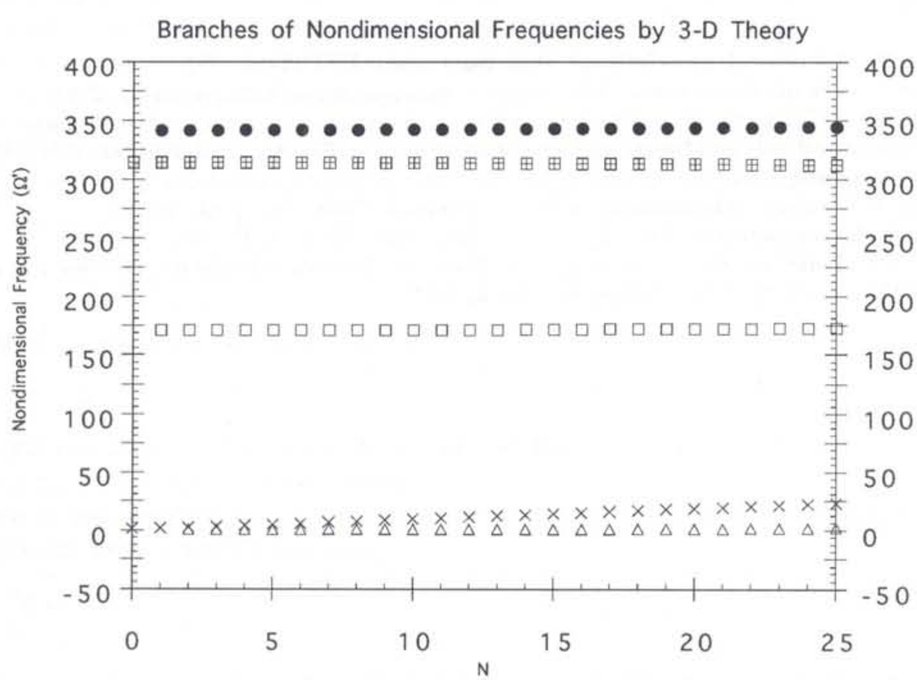
Branches of Nondimensional Frequencies by 3-D Theory
Nondimensional Frequency (Ω')
N

Fig. 9

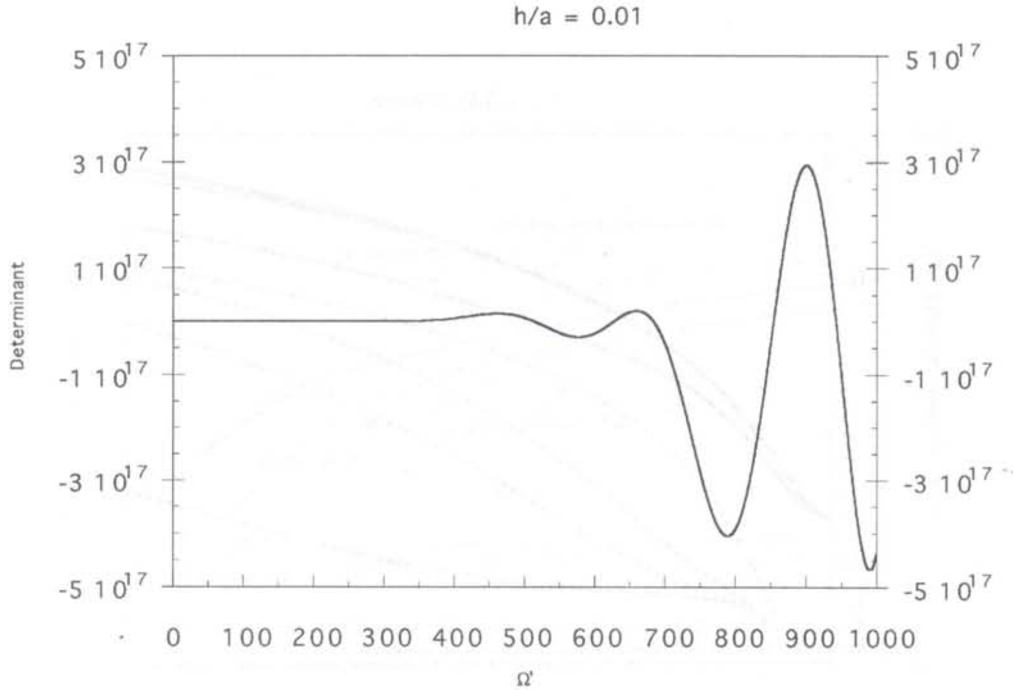

Fig. 10

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