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Application of the boundary element method to modelling of crack propagation trajectories

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The present paper further develops the boundary element technique to provide an efficient and accurate method of analysing the crack propagation processes in 2-D linear elastic structures. Based on both the direct boundary integral equations, for source points located on the external boundary of the plane elastic region, and the indirect boundary integral equations, for the resultant forces acting on one side of the crack surfaces, this technique allows to avoid problems associated with other numerical methods for fracture mechanics computation. In the first part of this paper, the proposed boundary element technique and also the strain energy density criterion, which determines the crack increment in a mixed-mode loading situations, are described. In the second part, two numerical examples are enclosed to demonstrate the capabilities of the boundary element technique as a tool for modelling an arbitrary crack, predicting its growth and updating the model geometry to simulate the next crack increment.

1. Introduction

The boundary element method (BEM), namely a general discretization procedure of continuum problems posed by the boundary integral equation method (BIE), is now firmly established in many engineering disciplines as an alternative numerical technique to "domain"-type solutions, such as the finite element method (FEM) and the finite difference method (FDM). The boundary element method has some important advantages over the finite element method, in particular for a boundary element analysis only surface meshes are required. This fact leads to more efficient computer codes and a more convenient preparation of the input data. If a finite element analysis is performed, the volume of the model must also be discretized (see [4]). One of the fields, where the BIE formulations were applied successfully, is linear elastic fracture mechanics.

Linear elastic fracture mechanics (LEFM) provides a theoretical basis (see [2]) for the prediction of fracture processes and the fracture-resistant design of many structures. The key feature, which was recognized by Irwin [21] and became the foundation of LEFM, is the fact that for limited plasticity the stress intensity factor k (SIF) characterizes the state of singular material loading over a sufficient size volume in the region around the crack tip. The most difficult problem in boundary and finite fracture modelling is the need to approximate the size of the elastic singular field near the crack tip, which is not known a priori for a particular problem. The extent of the singular fields is set by using special crack tip elements in the automatic selection of the modelling grid.

In modelling of crack propagation trajectories numerical complications arise resulting not only from the need to capture the singularities at the crack tips, but also from nonlinear responses caused by the opening/closing of fractures. When crack propagation is modelled, the changing crack geometry requires the element mesh of the crack model to be continuously modified. Automatic local re-meshing is used to simulate crack growth. Such re-meshing, at every propagation step, renders the "domain"-type solutions inefficient.

Straightforward application of the boundary element method to crack problems leads to a mathematical degeneration in the numerical formulation (i.e. the singular matrix) because the BIE formulations degenerate for a body with two surfaces occupying the same location in the form of a crack. To avoid this difficulty, several modelling strategies have been suggested. The first widely applicable strategy, of dealing with two coplanar crack surfaces, was devised by Blandford, Ingraffea and Liggett [5]. Their approach, known as the multi–domain formulation, is based on cutting the finite medium along the crack surfaces occupying the same location into two subregions. Then these subregions can be added, utilizing continuity and equilibrium relations at the cut interfaces. Note that the multi–domain formulation is, as yet, the most general. It can be applied to both symmetrical and non-symmetrical crack problems in both two- and three-dimensional configurations. However, this approach becomes inefficient when there are two or more cracks, or even for a single crack that propagates under mixed–mode loading. In addition, it introduces along the cut new unknowns that can significantly increase the size of the problem to be solved. Snyder and Cruse [33] developed the second strategy for dealing with two coplanar crack surfaces, which is called the special Green's function formulation. The fundamental solution (i.e. the Green's function) contains the exact form of the traction–free crack in an infinite medium, hence no modelling of the crack surfaces is required. In spite of the fact that the special Green's function technique is accurate, this technique is only restricted to the 2-D (two-dimensional) straight cracks approximations. Therefore, it cannot be applied to modelling the crack propagation trajectories. The third strategy of eliminating the problem of the crack BIE degeneration is the displacement discontinuity method. An extensive consideration of the displacement discontinuity method was given by Crouch and Starfield [9]. In this manner, the crack is directly treated as a single surface across which the displacements are discontinuous. Therefore, additional integral expressions for the crack surface stresses must be derived. Such expressions can be obtained by means of the Hooke's law. It must be noted that these integral expressions contain high singularities termed hyper-singularities. Hyper-singularities are very difficult to handle in numerical calculations. So far, for crack propagation problems the displacement discontinuity method seems to be very promising. Recently, the displacement discontinuity method was modified to the dual boundary element method (DBEM). This approach, which is based on two different types of equations (i.e. the displacement and traction equations) for coincident crack nodes, was introduced in [27,28,29] to account for 2-D and 3-D static fracture problems.

In this paper the application of the boundary element method to modelling of crack propagation trajectories is presented. The analysis, considered here, embraces two levels, i.e. the elementary crack growth increment in the direction of the minimum strain energy density factor and the crack propagation into the solid. The models of crack propagation trajectories have been determined using an incremental description. Numerical examples for the computation of crack paths are included.

2. Strain energy density criterion

In many practical situations structures are subjected to shear and tensile loads, which will lead to mixed-mode cracking. So far, mixed–mode crack propagation criteria are concerned under the LEFM assumptions. Three commonly known criteria can be summarized as follows:

- the maximum tensile stress factor (see [12]),
- the maximum energy release rate (see [20]),
- the minimum strain energy density factor (see [32]).

Based on a survey of experimental and numerical results (see [4,8]), all the three methods have been found to be satisfactory for crack trajectory prediction. However, there is no preferred choice for critical load prediction. Within the general framework of the variational principles, a physical interpretation of the strain energy density criterion can be formulated.

It is natural to explore the energy balance relation for an elastic cracked body. According to the conservation law, energy can be converted from one form to another, but the total quantity stays the same, i.e.

$$E = D + E_{\mathrm{k}} + E_{\mathrm{p}} = \mathrm{const}\,, \tag{1}$$

where D is the energy dissipated in the material, $E_{\rm k}$ is the kinetic energy and $E_{\rm p}$ is the potential energy. For limited plasticity $D \to 0$. The total potential energy of the mechanical system consists of the potential energy of the internal loads, U, and the potential energy of the external loads, W,

$$E_{\rm p} = U + W\,. \tag{2}$$

The potential energy of the internal loads is also termed the strain energy. If the simplified physical model of quasi-static crack growth is employed, then $E_{\rm k} = 0$ and equation (1) takes on the form

$$E = E_{\rm p} = U + W = \text{const}\,. \tag{3}$$

Moreover, the potential energy of the external loads is equal to the negative of the strain energy. Obviously, for the displacements corresponding to the state of equilibrium, the variation of the total potential energy is

$$\delta E_{\rm p} = \delta(U + W) = 0\,. \tag{4}$$

The above statement means that for the equilibrium to be ensured, the total potential energy must be stationary for variations of admissible displacements. In general, any change of total potential energy caused by an arbitrary deviation from the state of equilibrium can be expressed mathematically as the sum of variations

$$\Delta E_{\rm p} = \delta E_{\rm p} + \delta^2 E_{\rm p} + \delta^3 E_{\rm p} + \dots\,. \tag{5}$$

Assume that the external loads are independent of geometrically permissible displacements, such that the potential energy of the external loads is a linear function of displacements and the variations of this potential energy do not occur in powers exceeding one. Therefore, equation (5) due to the principle of a stationary value of the total potential energy (i.e. $\delta E_{\rm p} = 0$) reduces to the form

$$\Delta E_{\rm p} = \delta^2 U + \dots\,. \tag{6}$$

In stable elastic situations, the total potential energy is not only stationary but is a minimum by reason of the following relations

$$E_{\rm p} = \min \quad \text{if} \quad \delta E_{\rm p} = 0 \ \wedge \ \Delta E_{\rm p} = \delta^2 U > 0\,. \tag{7}$$

Let a deformable cracked body, which is in equilibrium, be subdivided arbitrarily. Let one of these divisions near the crack tip fill out volume V, which is bounded by surface S. Then the division, the same as the entire body, is also in equilibrium. Irwin [22] showed that the stress and displacement states in close neighbourhood of the smooth internal boundary of a plane crack in a linearly elastic solid under most general loading conditions, may be expressed in terms of three stress intensity factors k_1, k_2 and k_3 associated with the symmetric opening, in-plane and anti-plane shear modes of deformation, respectively. If the displacements for the whole surface S of the division are specified and no volume forces are present, then the strain energy assumes a stationary value in the class of geometrically permissible displacements for the true displacements corresponding to the state of equilibrium. When applied to Hooke's bodies, the stationary value of the strain energy is especially a minimum. For an elastic material the local strain energy $\mathrm{d}U$, which is stored in a volume element $\mathrm{d}V$, can be written in terms of the stress field components σ_{ij} as

$$\frac{\mathrm{d}U}{\mathrm{d}V} = \frac{1}{2E}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2\right) - \frac{\nu}{E}\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right) + \frac{1+\nu}{E}\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\,, \tag{8}$$

where E is the Young's modulus and ν is the Poisson's ratio. From the expressions for the asymptotic stress fields given in [14], the strain energy per unit volume near the crack tip may be expressed in the following form

$$\frac{\mathrm{d}U}{\mathrm{d}V} = \frac{1}{r_{\rm c}}\left(a_{11}k_1^2 + 2a_{12}k_1k_2 + a_{22}k_2^2 + a_{33}k_3^2\right) + \dots = \frac{S(\theta)}{r_{\rm c}} + \dots\,, \tag{9}$$

in which r_c is the distance from the crack tip and the coefficients, a_{ij}, can be found in [32]. The strain energy density function, dU/dV, becomes singular as $r_c \to 0$. Consequently, the strain energy density factor $S(\theta)$, which varies with the polar angle θ, is defined only if $r_c \neq 0$.

The strain energy density criterion states that quasi-static crack growth takes place in the direction of minimum strain energy density factor $S(\theta)$. The necessary and sufficient conditions for $S(\theta)$ to be a minimum are given by

$$\frac{\partial S(\theta)}{\partial \theta} = 0 \wedge \frac{\partial S^2(\theta)}{\partial \theta} > 0 \quad \text{at} \quad \theta = \theta_0 \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right) . \tag{10}$$

3. Analytical formulation and numerical procedure

3.1. Basic boundary integral equations

Before turning to the specification of the boundary integral equations, for clarity some notations will first be introduced (see Fig. 1). Let Γ_b denote the positively oriented external boundary of

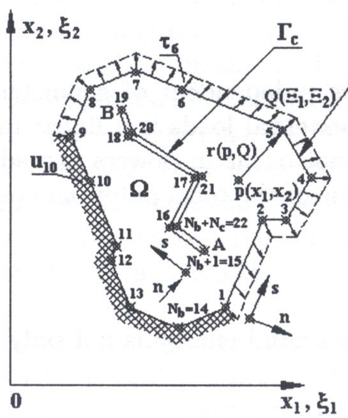

Fig. 1. Notations

the plane elastic region Ω. Let the region Ω contain a negatively oriented internal crack line Γ_c. The upper crack line is denoted by Γ_c^+ and the lower one is denoted by Γ_c^-. The part of $\Gamma = \Gamma_b + \Gamma_c$, where the displacements u_j are known, is marked by Γ_1 and the one, where the tractions τ_j are known, is marked by Γ_2. $r = r(p,q)$ is the distance between the two points p and q. In the following, the lower case symbols $p = (x_1, x_2)$ and $q = (\xi_1, \xi_2)$ represent the source and field points in the interior of the domain Ω, while the upper case symbols $P = (X_1, X_2)$ and $Q = (\Xi_1, \Xi_2)$ represent the source and field points located on the boundary Γ. Further, the components of a unit outward normal to Γ, at a point on it, are denoted by n_i, and s is the distance measured along Γ from some fixed point on it.

The usual 2-D BIE for the geometry in Fig. 1, in the absence of body forces, takes on the form

$$P \in \Gamma \quad \to \quad C_{ij}(P)u_j(P) = \oint_{\Gamma_1} U_{ij}(P,Q)\tau_j(Q)\mathrm{d}\Gamma(Q) - \oint_{\Gamma_2} T_{ij}(P,Q)u_j(Q)\mathrm{d}\Gamma(Q) \tag{11}$$

in which C_{ij} are the coefficients depending on the local geometry of $\mathrm{d}\Gamma$ at P. $U_{ij}(p,q)$ is called the fundamental (or Kelvin's) solution and represents the displacement $u_i^*(p,q)$ in the direction i at any field point q caused by the unit force e_j, applied at the load point p in the direction j in an infinite elastic medium. This fundamental displacement solution may be redefined in terms of the second-order tensor $U_{ij}(p,q)$ as $u_i^*(p,q) = U_{ij}(p,q)e_j$. The fundamental solution $T_{ij}(p,q)$ is the traction $\tau_j^*(p,q)$ (or, owing to the fact that $\tau_i = \sigma_{ij}n_j$, the stress vector related to the normal vector n_j) in the direction i at any field point q due to the unit force e_j, applied at the load point p in the direction j, i.e. $\tau_i^*(p,q) = T_{ij}(p,q)e_j$. Explicit expressions corresponding to the Kelvin's solution (see [6,26]) of elastostatics are given for 2-D plane strain conditions by

$$U_{ij} = \frac{-1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln(r)\,\delta_{ij} - r_{,i}r_{,j}\right], \tag{12a}$$

$$T_{ij} = \frac{-1}{4\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}\right] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right], \tag{12b}$$

where $\mu = E/[2(1+\nu)]$ is the shear modulus of elasticity and δ_{ij} is the Kronecker delta.

The fundamental problem in using equation (11), for fracture mechanics applications, is seen as $\Gamma_c^+ \to \Gamma_c^-$. If the tractions on the two crack lines, Γ_c^+ and Γ_c^-, are equal and opposite (i.e. $\tau_i(Q^+) = \tau_i(Q^-)$), then from expressions (12) it can be shown that $T_{ij}(P,Q^+) = -T_{ij}(P,Q^-)$ and $U_{ij}(P,Q^+) = U_{ij}(P,Q^-)$ for $Q \in \Gamma_c$. Owing to these facts, equation (11) for $P \in \Gamma_b$ can be written in the form

$$P \in \Gamma_b \quad \to \quad C_{ij}u_j = \oint_{\Gamma_{b1}} U_{ij}\tau_j \mathrm{d}\Gamma_b - \oint_{\Gamma_{b2}} T_{ij}u_j \mathrm{d}\Gamma_b + \int_{\Gamma_c^-} T_{ij}\Delta u_j^c \,\mathrm{d}\Gamma_c^-, \tag{13}$$

where $\Delta u_j^c = u_j(Q^+) - u_j(Q^-)$. Equation (13) is commonly referred to as the displacement discontinuity method for crack problems. In equation (13) the unknown functions on Γ_c^- are the displacement differences between the upper and lower crack lines. However, if the source point P is located on the crack line Γ_c^-, equation (13) is obviously indeterminate. In this case, there are two unknown displacement variables on Γ_c^-, namely Δu_j^c and u_j^c. Therefore, an additional integral expression for the crack line must be derived.

Assume that Γ_b and Γ_c^- have no the common point. For an isotropic medium the stresses at the internal point p can be calculated by differentiating u and using the Hooke's law, written in the form

$$\sigma_{ij} = \frac{2\nu\mu}{1-2\nu}\delta_{ij}\frac{\partial u_l}{\partial x_l} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{14}$$

After differentiation, expression (14) becomes

$$\begin{aligned} p \in \Omega \quad \to \quad \sigma_{ij} = &\oint_{\Gamma_{b1}} \left\{\frac{2\nu\mu}{1-2\nu}\delta_{ij}\frac{\partial U_{lk}}{\partial x_l} + \mu\left(\frac{\partial U_{ik}}{\partial x_j} + \frac{\partial U_{jk}}{\partial x_i}\right)\right\}\tau_k \,\mathrm{d}\Gamma_b \\ &- \oint_{\Gamma_{b2}} \left\{\frac{2\nu\mu}{1-2\nu}\delta_{ij}\frac{\partial T_{lk}}{\partial x_l} + \mu\left(\frac{\partial T_{ik}}{\partial x_j} + \frac{\partial T_{jk}}{\partial x_i}\right)\right\}u_k \,\mathrm{d}\Gamma_b \\ &+ \int_{\Gamma_c^-} \left\{\frac{2\nu\mu}{1-2\nu}\delta_{ij}\frac{\partial T_{lk}}{\partial x_l} + \mu\left(\frac{\partial T_{ik}}{\partial x_j} + \frac{\partial T_{jk}}{\partial x_i}\right)\right\}\Delta u_k^c \,\mathrm{d}\Gamma_c^-, \end{aligned} \tag{15}$$

or

$$p \in \Omega \quad \to \quad \sigma_{ij} = \oint_{\Gamma_{b1}} D_{ijk}\tau_k \,\mathrm{d}\Gamma_b - \oint_{\Gamma_{b2}} S_{ijk}u_k \,\mathrm{d}\Gamma_b + \int_{\Gamma_c^-} S_{ijk}\Delta u_k^c \,\mathrm{d}\Gamma_c^-, \tag{16}$$

where

$$D_{ijk} = \frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} - \delta_{ij}r_{,k}) + 2r_{,i}r_{,j}r_{,k}\right], \tag{17a}$$

$$\begin{aligned} S_{ijk} = \frac{2\mu}{4\pi(1-\nu)r^2}&\left\{2\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}\right]\right. \\ &\left. + 2\nu(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}) + (1-2\nu)(2n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}) - (1-4\nu)n_k\delta_{ij}\right\}. \end{aligned} \tag{17b}$$

Let the outer boundary Γ_b in Fig. 1 be remote and unloaded. Then, expression (16) reduces to the form

$$p \in \Omega \quad \rightarrow \quad \sigma_{ij}^{c}(p) = \int_{\Gamma_c^-} S_{ijk}(p, Q^-) \Delta u_k^c(Q^-) \, d\Gamma_c^-(Q^-) , \tag{18}$$

which leads to setting the crack problems in an infinite plane elastic region. For $p \rightarrow P \in \Gamma_c^-$ (i.e. $r \rightarrow 0$) the integral kernel $S_{ijk}(p, Q^-)$ in expression (18) has a fundamental singularity, which is generally called hyper-singularity of the order $O(r^{-2})$. This hyper-singularity renders the use of expression (18) as a boundary integral equation difficult. However, Cruse [10], Guidera and Lardner [16], Bui [7], Weaver [37], Balas and Sladek [3], Theocaris [36], and Putot [30] showed that the hyper-singularity in expression (18) can be reduced by one order by means of integration by parts. Takakuda, Koizumi and Shibuya [34] proposed a technique of integration based on the Taylor's series expansion of $\Delta u_j^c(Q)$ about the point Q on the crack surface. They treated the resulting integrals in Hadamard's [18] finite part sense. More recently, by using also the Taylor's series expansion, Krishnasamy, Rizzo, Rudolphi and Schmerr [25,31] developed another method for the treatment of hyper-singular integrals. They subtracted and added a number of terms of the Taylor's series expansion of the integrand at the singular point in the manner proposed by Davis and Rabinowitz [11], and Guiggiani and Casalini [17].

It is worthwhile to stress that there is the possibility of coupling the two equations (13) and (16) in order to attain the numerical solution, which allows us to determine the unknown displacements u_j^c on the upper and lower crack lines (see [19]). If the crack problem in an infinite medium is considered, then different boundary integral equations on either smooth crack line are given by

$$\begin{aligned} P \in \Gamma_c^+ \quad \rightarrow \quad & \frac{1}{2} u_i^c(P^+) + \frac{1}{2} u_i^c(P^-) + \int_{\Gamma_c^+} T_{ij}(P, Q^+) u_j^c(Q^+) \, d\Gamma_c(Q^+) \\ & - \int_{\Gamma_c^+} U_{ij}(P, Q^+) \tau_j^c(Q^+) \, d\Gamma_c(Q^+) = 0 , \end{aligned} \tag{19a}$$

$$\begin{aligned} P \in \Gamma_c^- \quad \rightarrow \quad & \frac{1}{2} \tau_i^c(P^-) - \frac{1}{2} \tau_i^c(P^+) + \int_{\Gamma_c^-} G_{ij}(P, Q^-) u_j^c(Q^-) \, d\Gamma_c(Q^-) \\ & - \int_{\Gamma_c^-} T_{ij}^T(P, Q^-) \tau_j^c(Q^-) \, d\Gamma_c(Q^-) = 0 , \end{aligned} \tag{19b}$$

where $G_{ij} = n_\alpha S_{\alpha ij}$ and $T_{ij}^T = n_\alpha U_{\alpha ij}$. The kernel U_{ij} is weakly singular of the order $O(\ln(r))$, the kernels T_{ij} and T_{ij}^T are strongly singular of the order $O(r^{-1})$, and the kernel G_{ij} is hyper-singular of the order $O(r^{-2})$.

Taking into account the ideas developed by Ghosh et al. [15] and Zang [38], the author concentrates on the following boundary integral equations

$$P \in \Gamma_b \quad \rightarrow \quad C_{ij} u_j = \oint_{\Gamma_{b1}} U_{ij} \tau_j \, d\Gamma_b - \oint_{\Gamma_{b2}} T_{ij} u_j \, d\Gamma_b + \int_{\Gamma_c^-} W_{ij} \frac{\partial}{\partial s^-} \left[\Delta u_j^c \right] ds^- , \tag{20a}$$

$$P \in \Gamma_c^- \quad \rightarrow \quad F_i^- = \oint_{\Gamma_{b1}} F_{ij}^\tau \tau_j \, d\Gamma_b + \oint_{\Gamma_{b2}} F_{ij}^{u} u_j \, d\Gamma_b + \int_{\Gamma_c^-} F_{ij}^{d} \frac{\partial}{\partial s^-} \left[\Delta u_j^c \right] ds^- + C_i . \tag{20b}$$

where F_i^- are the resultant forces acting along the lower side of the crack lines (i.e. Γ_c^-), $\partial / \partial s^- [\Delta u_j^c]$ are the dislocation densities along Γ_c^-, C_i are arbitrary constants and W_{ij}, F_{ij}^τ, F_{ij}^{u}, F_{ij}^{d} are the integral kernels. Explicit representations of W_{ij}, F_{ij}^τ, F_{ij}^{u} and F_{ij}^{d} are given by

$$W_{ij} = \frac{1}{4\pi(1-\nu)} \left[2(1-\nu)\phi\delta_{ij} + \epsilon_{jk} r_{,i} r_{,k} + (1-2\nu)\epsilon_{ij} \ln(r) \right] , \tag{21}$$

$$F_{ij}^{\tau} = \frac{-1}{4\pi(1-\nu)}\left[2(1-\nu)\phi\delta_{ij} + \epsilon_{ik}r_{,j}r_{,k} - (1-2\nu)\epsilon_{ij}\ln(r)\right] , \tag{22a}$$

$$F_{ij}^{\mathrm{u}} = \frac{-\mu}{2\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}r_{,k}(r_{,i}\epsilon_{jk} + r_{,j}\epsilon_{ik}) + (r_{,2}n_1 - r_{,1}n_2)\delta_{ij}\right] , \tag{22b}$$

$$F_{ij}^{\mathrm{d}} = \frac{-\mu}{2\pi(1-\nu)}\left[\ln(r)\delta_{ij} - r_{,i}r_{,j}\right] , \tag{22c}$$

in which $\tan\phi = (\xi_2 - x_2)/(\xi_1 - x_1)$ and ϵ_{ij} is the ϵ-tensor. The kernels W_{ij}, F_{ij}^{τ}, and F_{ij}^{d} are weakly singular of the order $O(\ln(r))$, and the kernel F_{ij}^{u} is strongly singular of the order $O(r^{-1})$. Application of equations (20) makes it possible to avoid the numerical difficulties with the hyper-singular integrals.

In general, a contour of the crack line Γ_{c} may be closed or open. Thus, two types of cracks may be distinguished, i.e.

- an internal crack with a closed contour of Γ_{c},
- an edge crack with an open contour of Γ_{c}.

For an internal crack, the following constraint equation must be fulfilled

$$\int_{\Gamma_{\mathrm{c}}^{-}} \frac{\partial}{\partial s^{-}}\left[\Delta u_j^{\mathrm{c}}\right] \mathrm{d}s^{-} = 0 . \tag{23}$$

Straightforward application of equations (20) to edge crack problems gives invalid results. For this reason, there were attempts at using these equations with integral kernels formulated on the basis of the half-plane fundamental solutions (see [38]). The integral kernels formulated in that way contain two parts. The first one is the same as that based on the infinite plane fundamental solutions. The other is a complementary part to the first one due to the presence of the half-plane. In addition, the boundary conditions on a straight line, which is the part of the external boundary Γ_{b} and represents the surface of the half-plane, are automatically satisfied with these kinds of kernels. Thus, this line has not to be discretized in the numerical calculations, thus considerably reducing the amount of numerical work involved in the solution of the problem. However, the approach based on the half-plane fundamental solution is rather limited to surface crack problems, e.g. an edge crack originating from the straight linear boundary or an internal sub-surface crack close to the straight linear boundary with prescribed tractions. Note that using this approach, the additional constraint equation (23) is invalid for edge crack problems.

A more general method of edge crack modelling is the method based on the point symmetry conditions for the region very close to an edge crack. Such point symmetry conditions are established in the numerical way and satisfied the continuity and equilibrium relations for the introduced additional part of the external boundary Γ_{b}. A simple example of the edge crack problem, shown in Fig. 2, illustrates the idea of the author's approach. This approach will be explained in the next author's paper.

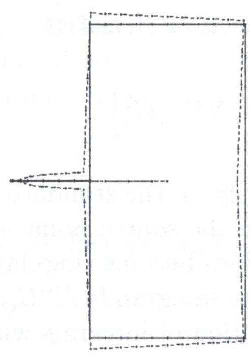

Fig. 2. Discretization of an edge cracked plate under tension

3.2. Numerical treatment

Analytical solution of the BIE formulation in the form of equations (20) is not generally possible. Therefore, a numerical solution is required. A simple numerical treatment of equations (20) is presented here, in which the external boundary is approximated by N_b straight elements, and the crack line Γ_c^- by N_c^- straight elements (note that $N_\mathrm{c} = N_\mathrm{c}^+ + N_\mathrm{c}^-$), as shown in Fig. 1. Then equations (20) can be written as

$$P \in \Gamma_\mathrm{b} \quad \rightarrow \quad C_{ij}u_j = \sum_{n=1}^{N_\mathrm{b}} \oint_{\Gamma_n} [U_{ij}\tau_j - T_{ij}u_j]\,\mathrm{d}\Gamma_n + \sum_{n=N_\mathrm{b}+1}^{N_\mathrm{b}+N_\mathrm{c}^-} \int_{\Gamma_n} W_{ij}\frac{\partial}{\partial s^-}\big[\Delta u_j^\mathrm{c}\big]\,\mathrm{d}\Gamma_n\,, \tag{24a}$$

$$P \in \Gamma_\mathrm{c}^- \quad \rightarrow \quad F_i^- = \sum_{n=1}^{N_\mathrm{b}} \oint_{\Gamma_n} \Big[F_{ij}^{\tau}\tau_j + F_{ij}^{\mathrm{u}}u_j\Big]\,\mathrm{d}\Gamma_n + \sum_{n=N_\mathrm{b}+1}^{N_\mathrm{b}+N_\mathrm{c}^-} \int_{\Gamma_n} F_{ij}^{\mathrm{d}}\frac{\partial}{\partial s^-}\big[\Delta u_j^\mathrm{c}\big]\,\mathrm{d}\Gamma_n + C_i\,, \tag{24b}$$

where $\Gamma = \sum_{n=1}^{N_\mathrm{b}+N_\mathrm{c}^-} \Gamma_n$.

On each element the displacement field u_j, the traction field τ_j and the dislocation densities $G_j = \partial/\partial s^-[\Delta u_j^\mathrm{c}]$ can be approximated by interpolation functions in the following manner:

$$u = \sum_{\alpha=1}^{m} N^\alpha u^\alpha\,, \tag{25a}$$

$$\tau = \sum_{\alpha=1}^{m} N^\alpha \tau^\alpha\,, \tag{25b}$$

$$G = \sum_{\alpha=1}^{m} N^\alpha G^\alpha\,, \tag{25c}$$

in which N^α are called shape functions. The shape functions are polynomials of degree $m-1$ and have the property that they are equal to one at node α and zero at all other nodes. u^α, τ^α, and G^α are the values of the functions at node α. The shape functions are defined in terms of the non-dimensional coordinate ζ $(-1 \le \zeta \le 1)$. For boundary elements Γ_n, the shape functions approximated linearly (i.e. $m = 2$) become

$$N^1(\zeta) = \frac{1-\zeta}{2}\,, \tag{26a}$$

$$N^2(\zeta) = \frac{1+\zeta}{2}\,. \tag{26b}$$

However, for the elements which contain the crack tip at node $\alpha = 1$ or $\alpha = 2$, the following interpolation has to be used for the dislocation densities

$$G_j^{\mathrm{c}(\alpha=1/\alpha=2)}(\zeta) = \sqrt{\frac{2}{1 \pm \zeta}}\,\Big[N^1(\zeta)G_j^1 + N^2(\zeta)G_j^2\Big]\,. \tag{27}$$

Current BEM codes rely almost entirely on the standard Gauss-Legendre integration scheme to perform regular integrals (see [6]). When the source point is located within the boundary element limits, element integrals of BIE formulations become singular. General application of equations (24) requires computation of integrals with the integrands $N^\alpha U_{ij}$, $N^\alpha W_{ij}$, $N^\alpha F_{ij}^{\tau}$, $N^\alpha F_{ij}^{\mathrm{d}}$ which are singular of the order $O(\ln(r))$ and computation of integrals with the integrands $N^\alpha T_{ij}$, $N^\alpha F_{ij}^{\mathrm{u}}$ which are singular of the order $O(r^{-1})$. Moreover, if the boundary elements Γ_n contain the crack tip, then the integrand $N^\alpha F_{ij}^{\mathrm{d}}$ is singular of the order $O(\ln(r)r^{-1/2})$. The basic question to be answered is how should we integrate the singular integrals with respect to r? The singular integrals for linear elements may be computed analytically or numerically. The numerical calculation of the singular integrals can be carried out in two different ways. The first consists in separating the integral kernel into a regular finite part (to be integrated by the Gauss-Legendre integration scheme) and a dependent part, which can be integrated by a special numerical scheme with weighting coefficients (see [6]). Because of a boundary element partition, such procedure can be too cumbersome in some cases. The second alternative relies on a suitable coordinate transformation, its Jacobian smoothing out the singularity. The nonlinear coordinate transformation, which was developed by Telles [35], automatically lumps the integration points towards the minimum source distance position without the disadvantage of subdividing the boundary element. Therefore, this scheme of numerical integration can easily be set up into existing BEM codes which use the Gauss-Legendre integration. This scheme can be explained as follows.

Consider the integral

$$I = \int_{-1}^{1} f(\eta)\,\mathrm{d}\eta\,, \tag{28}$$

in which $f(\eta)$ is singular at a point $\bar{\eta}$. If a second-degree relation

$$\eta(\gamma) = a\gamma^2 + b\gamma + c \tag{29}$$

satisfies the following requirements

$$\begin{cases} \left.\dfrac{\mathrm{d}\eta}{\mathrm{d}\gamma}\right|_{\bar{\eta}} & = & 0 \\ \eta(1) & = & 1 \\ \eta(-1) & = & -1 \end{cases} \tag{30}$$

then for $|\bar{\eta}| = 1$ expression (28) takes on the form

$$I = \int_{-1}^{1} f\left[(1-\gamma^2)\frac{\bar{\eta}}{2} + \gamma\right](1-\gamma\bar{\eta})\,\mathrm{d}\gamma\,. \tag{31}$$

The above transformation can be used to calculate integrals with a logarithmic singularity at one of the extremities. Application of a third-degree polynomial transformation allows to improve the accuracy of this integration scheme within-the near-singularity range. The results, obtained by the author [23], show that the Gauss–Legendre integration with the self-adaptive coordinate transformation can be used to solve equations (24). Note that other methods based on numerical quadrature are also available for crack BIE formulations, e.g. Gauss–Chebyshev quadrature developed by Erdogan, Gupta, and Cook [13], and Labotto–Chebyshev method (see [24]).

In this approach, straight boundary elements are used. Consequently, curved boundaries can be approximated by a series of straight elements.

3.3. SIF COMPUTATION

Using the 2-D BEM technique, which was described in Section 3.2, it is possible to evaluate the dislocation densities G_j at each crack nodal point. The dislocation densities normal and tangent to the crack surfaces at the nodal point are given by

$$G_\mathrm{n} = G_1 n_1 + G_2 n_2\,, \tag{32a}$$

$$G_\mathrm{t} = G_2 n_1 - G_1 n_2\,. \tag{32b}$$

Thus, the non-dimensional stress intensity factors can be found from

$$K_\mathrm{I} = \frac{2\mu}{\kappa+1}\sqrt{2\pi d}\,G_\mathrm{n}^\mathrm{c}\,, \tag{33a}$$

$$K_\mathrm{II} = \frac{2\mu}{\kappa+1}\sqrt{2\pi d}\,G_\mathrm{t}^\mathrm{c}\,, \tag{33b}$$

where d is the length of the element close to the crack tip, G_n^c and G_t^c are the nodal values at the crack tip, $\kappa = (3 - 4\nu)$ for plane strain condition and $\kappa = (3-\nu)/(1+\nu)$ for plane stress condition.

4. Application examples

The ideas and formulations given in the previous sections have been incorporated into the computer code developed by the author. Various case studies have been performed to verify and test the capability of the computer program (see [23]). Two examples from these case studies are presented in the following sections. The presented numerical results were carried out on the IBM AT 386 computer.

4.1. Modelling of crack propagation originating from a sub-surface crack in a rectangular plate

The problem of development of a planar crack propagating from an initially straight sub-surface crack in a rectangular plate is considered. Fig. 3 shows the model of the rectangular plate containing

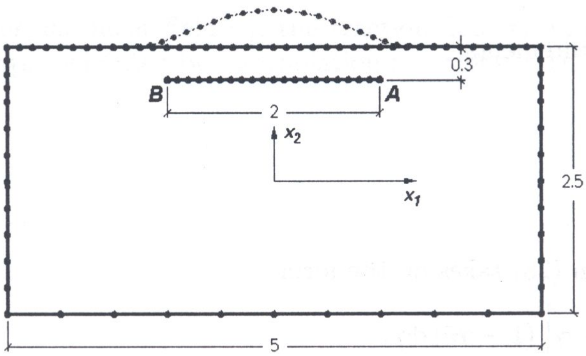

Fig. 3. Discretization of the plate with the sub-surface crack

the sub-surface crack. The external boundary element mesh consists of 74 linear elements. Moreover, the initial lower crack line is discretized into 28 linear elements. The sub-surface crack is subjected to a uniform pressure p_0, which is normal to its surfaces. In Fig. 3 the dashed line represents the deformed external boundary element mesh. Displacements of the deformed element mesh are magnified by a factor 2. Note that the material parameters are determined by $\nu = 0.3$ and $E/p_0 = 205$.

In Figs. 4 and 5 the relative crack surface displacements Δu_1 and Δu_2 are plotted along the initial crack profile. Readers can compare the present results, which are shown in Figs. 4 and 5, to the BEM results determined by Zang [38].

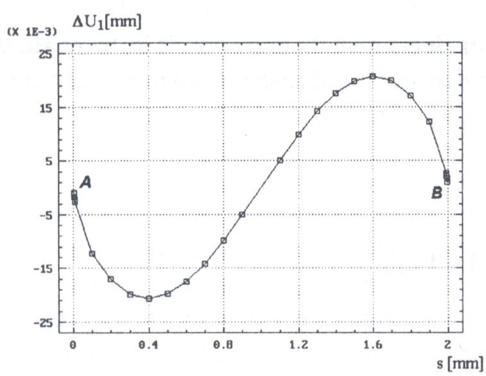

Fig. 4. Relative crack surface displacement Δu_1 along the sub-surface crack line

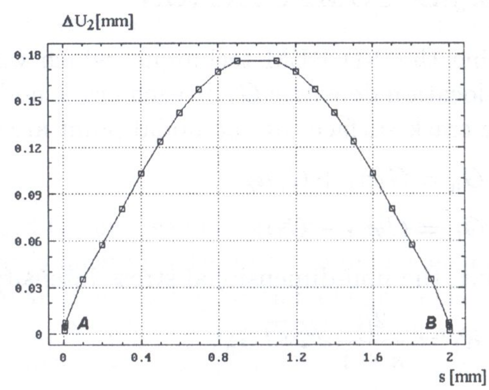

Fig. 5. Relative crack surface displacement Δu_2 along the sub-surface crack line

Figure 6 depicts the comparison between two boundary element meshes: the undeformed external mesh with the deformed one (displacements magnified by a factor of 2), which is influenced by the normal pressure p_0 distributed along both the initial crack line and the predicted crack trajectories. The magnified part of the expected crack trajectory is shown in Fig. 7.

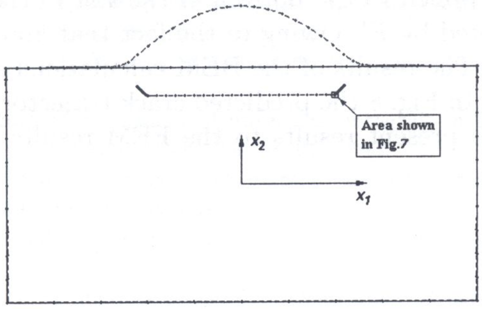

Fig. 6. Discretization of the plate with the sub-surface crack and predicted crack trajectories

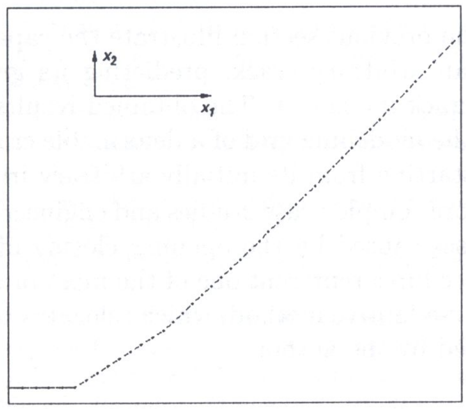

Fig. 7. Magnified near-kink mesh, of predicted crack trajectories, from Fig. 6

4.2. Modelling of crack propagation in a plate containing a rectangular slot

The problem of crack propagation modelling in a plate containing a rectangular slot is considered. The model of the plate is given in Fig. 8. The geometry and load are symmetric with respect to the

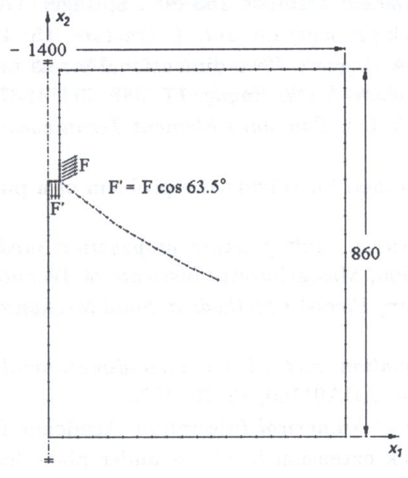

Fig. 8. Discretization of the plate with the rectangular slot and predicted crack trajectory

line bisecting the width of the plate. Due to this symmetry, only one half of the plate is modelled. The external boundary element mesh consists of 110 linear elements. The geometry and material characterization for this example are taken from [4]. Therefore, the depth of the slot is $h = 260$ mm and its width is $2a_0 = 57$ mm. The traction τ is distributed on its sides, over the distance $l_0 = 50$ mm from its bottom. The resultant force F of the traction τ is inclined at an angle of 63.5^0 with respect to x_2. Another traction τ' is applied to the bottom of the slot in the x_2– direction. The resultant force of the traction τ' is denoted by F'. Owing to the fact that an overall force equilibrium exists in the plate, $F' = F \cos 63.5^0$. The results of the BEM calculations are summarized in the form of the predicted crack trajectory. In Fig. 8 the predicted crack trajectory is represented by the dashed line. Readers can compare the present results to the FEM results determined by Bergkvist and Guex [4].

5. Conclusions

The boundary element method provided general–purpose numerical solutions of the 2-D elastic fracture mechanics problems, has been presented. This method has been applied to modelling of crack propagation trajectories.

The examples described in the previous section illustrate the capabilities of the 2-D BEM technique as a tool for modelling an arbitrary crack, predicting its growth and refining the model geometry to simulate the next crack increment. The obtained results show that this technique enables an automatic selection of the modelling grid of a deformable cracked body to follow the shape development of a planar crack starting from its initially arbitrary irregular profile.

Future papers will address more complex case studies and engineering applications. Among other problems, the nonlinear responses caused by the opening/closing of fractures will be considered. Additionally, since adaptive procedures represent one of the most promising paths for the improvement of the results (see [1]), the r-adaptive method, which relocates the nodes, will be implemented into the computer code developed by the author.

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