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Optimal design of reinforced concrete beams and frames

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Rectangular cross-sections of reinforced concrete beams and columns with nonsymmetric reinforcement are considered in the paper. The objective function represents the total cost of concrete, steel and formwork. Several dimensional and behavioral constraints (bearing capacity, cracking, deflection) are allowed for. The problem was formulated in general form so that introduction of specific regulations following from national codes is possible. The computer program for optimal design of beams and frames loaded in-plane has been developed. The numerical examples were computed taking into account the rules of Polish Design Code.

1. Introduction

Traditional way of designing r.c. structures consists of three separate actions: preselection of dimensions of structural elements, structural analysis and finally designing the reinforcement. If the initially assumed dimensions appear unsatisfactory, then the whole procedure is repeated. However, this method does not ensure optimal design. Therefore optimal design of r.c. structures has focused the attention of engineers in the last decade. General considerations about optimization of r.c. structures are presented in [4]. The derivation of optimality criteria for cross-sectional shape of post-stressed concrete beams was presented in [1]. Programming methods of optimal design were used in studies [2, 3, 5, 7]. In [2] the cross-sections of arbitrary shape were allowed for. Optimality criteria for r.c. beams and their use in optimal design was presented in [6, 8]. The present paper further extends the research on optimal design of r.c. structures allowing for various serviceability limit constraints in the formulation of the problem. The study is based on the general design philosophy of r.c. structures used in national codes and in Eurocode No. 2 [9]. The considerations are limited to beam and frame structures loaded in-plane. The above mentioned three separate design actions are integrated in one procedure of optimal design using nonlinear programming. The option for limited optimization, namely optimal design of reinforcement for preselected cross-sections of structural elements, is incorporated in the program. The numerical examples illustrate the engineering applicability of the method.

2. Formulation of the problem

Consider a beam or frame structure with prescribed support and joint conditions. Let it be subjected to multiple load sets. Similarly to [9], denote by b, h, d the width, depth and effective depth of a cross-section and by A_{s1}, A_{s2} the reinforcement areas in tension and compression, respectively. Assume that d follows explicitly from h, namely $d = h - 3$ cm. Our aim is to design optimally the dimensions of cross-sections b, h, and reinforcement A_{s1}, A_{s2} for all members of a frame, whereas the reinforcement is computed for three sections of each member, namely in midspan and at both ends. These eight cross-sectional parameters referred to all independent members of frame constitute the design vector **s**. For practical engineering reason, we assume the same width b for all structural members. Assume that the objective function represents the cost of a frame

$$\begin{aligned} F(\mathbf{s}) = {} & c_c b \sum_{i=1}^{NE} h_i l_i + c_s \sum_{i=1}^{NC} (A_{s1} + A_{s2})_i l_i \\ & + c_s \left(\sum_{i=1}^{NB} 0.7(A_{s1} + A_{s2}) l_i + 0.2(A_{s1}^{l} + A_{s2}^{l} + A_{s1}^{r} + A_{s2}^{r}) l_i \right) \\ & + c_f \sum_{i=1}^{NC} 2(b + h_i) l_i + c_f \sum_{i=1}^{NB} (b + 2h_i) l_i \, , \end{aligned} \tag{1}$$

where c_c, c_s and c_f denote the unit cost of concrete, steel and formwork, respectively. The superscripts ^{l} and ^{r} refer to the left and right support of a beam. The following limit numbers will be used in the paper:

NB = Number of Beams,
NC = Number of Columns,
NE = Number of structural Elements (NE=NB+NC),
NET = Number of independent Element Types (NET $\leq$ NE),
NJ = Number of Joints,
NDV = Number of dimensional Design Variables h_i.

The constraint for the limit state of bearing capacity, called in [9] *the ultimate limit state* is introduced using the limit interaction curves $\Phi(M, N) = 0$, which are computed for the actual value of design variable **s** and for all members of the frame. This constraint reads

$$\Phi(M, N) \leq 0 \, . \tag{2}$$

Here M and N denote bending moment and axial force, respectively (where $N > 0$ for compression). The stability is considered in the form of a constraint

$$(N_{sd})_i \leq \beta (N_{crit})_i \quad \text{for} \quad i = 1, 2, \ldots, NC \, , \tag{3}$$

where N_{crit} is evaluated according to [9] and β is user supplied coefficient, for example $\beta = 0.8$. We allow for the displacement constraints in beams

$$a_i \leq (a_{adm})_i \quad \text{for} \quad i = 1, 2, \ldots, NB \tag{4}$$

and crack width constraints in beams and columns

$$(w_k)_{ij} \leq w_{adm} \quad \text{for} \quad i = 1, 2, \ldots, NB; \; j = 1, 2, 3 \, , \tag{5}$$

$$(w_k)_{ij} \leq w_{adm} \quad \text{for} \quad i = 1, 2, \ldots, NC; \; j = 1, 2, 3 \, , \tag{6}$$

where: $j = 1$ — left support, $j = 2$ — span, $j = 3$ — right support.

Note that crack width in beams is constrained in the span and in regions of supports. The latter cracks result from shear. In order to limit the principal compressive stress in concrete, induced by shear force, the following constraint is introduced

$$(V_{Sd})_{ij} \leq (V_{Rd2}) \quad \text{for} \quad i = 1, 2, \ldots, NE; \; j = 1, 3 \, , \tag{7}$$

where V_{Sd} denotes the design shear force resulting from F.E. analysis, whereas V_{Rd2} denotes the maximum design shear force that can be carried without crushing the concrete. In the computer program $V_{Rd2} = 0.25bdf_{cd}$ was assummed [10]. Next constraints are of geometric type. The reinforcement ratio is limited by constraints (8)–(10)

$$\frac{(A_{s1})_i}{bd_i} \geq \rho_{min} \quad \text{for} \quad i = 1, 2, \dots, NE, \tag{8}$$

$$\frac{(A_{s2})_i}{bd_i} \geq \rho_{min} \quad \text{for} \quad i = 1, 2, \dots, NE, \tag{9}$$

$$\frac{(A_{s1})_i + (A_{s2})_i}{bd_i} \leq \rho_{max} \quad \text{for} \quad i = 1, 2, \dots, NE. \tag{10}$$

It is reasonable to constrain the ratio of stiffness of structural elements connected in one joint of a frame. Therefore we impose

$$\left[\frac{\left(k\dfrac{EI}{l}\right)_{max}}{\left(k\dfrac{EI}{l}\right)_{min}}\right] \leq 7 \quad \text{for} \quad i = 1, 2, \dots, NJ, \tag{11}$$

where $k = 4$ for clamped-clamped rods and $k = 3$ for clamped-hinged rods.

The dimensions of cross-sections are limited by constraints

$$1.0 \leq \frac{h_i}{b} \leq 3.5 \quad \text{for} \quad i = 1, 2, \dots, NDV, \tag{12}$$

$$\frac{l_{oi}}{h_i} \leq 30 \quad \text{for} \quad i = 1, 2, \dots, NC. \tag{13}$$

Additional geometric constraints, e.g. symmetry constraints for frames can easily be introduced. The reinforcement for shear and torsion is not considered within this formulation.

For computation of (2) in the ultimate limit state, the equilibrium state shown in Fig. 1 was assumed. The constitutive relations $\sigma = \sigma(\varepsilon)$ for concrete and steel can be adopted from [9] or [10].

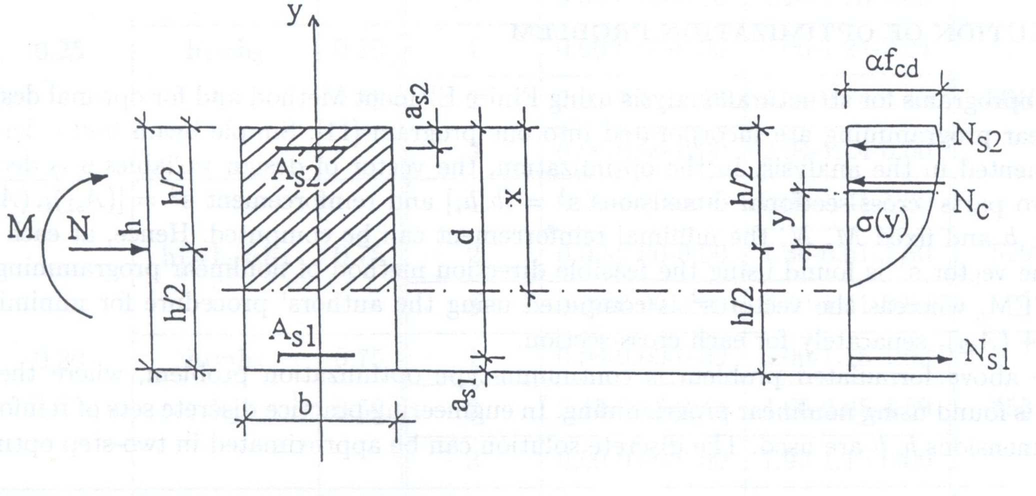

Fig. 1. Ultimate state of stress

The following equilibrium equations are used

$$N = b\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_c\left[\varepsilon(y)\right] dy + N_{s2} - N_{s1}, \tag{14}$$

$$M = b\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_c\left[\varepsilon(y)\right] y\, dy + N_{s2}(0.5h - a_{s2}) + N_{s1}(0.5h - a_{s1}). \tag{15}$$

Here M and N follow from linear finite element analysis. The bending moment in columns is increased allowing for random eccentricities, which are assumed as deterministic values [10]. An illustrative example of the limit interaction curve $\Phi(M, N) = 0$ computed from (14) and (15) for a rectangular cross-section unsymmetrically reinforced is shown in Fig. 2.

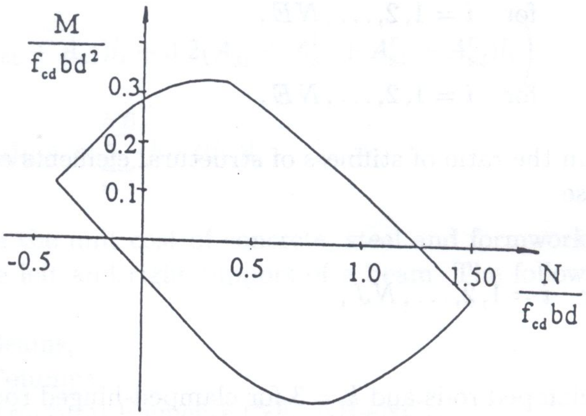

Fig. 2. Limit interaction curve for a rectangular cross-section and $A_{s1} < A_{s2}$

The flexural displacements a in (4) are approximated by multiplication of elastic displacements resulting from FE analysis by the stiffness ratio, namely

$$a = a_{\text{elastic}} \frac{EI}{B}, \tag{16}$$

where B denotes the stiffness of r.c. element in the cracked state.

3. Solution of optimization problem

The subprograms for structural analysis using Finite Element Method and for optimal design using nonlinear programming are incorporated into one program [11]. Simple beam finite elements are implemented in the analysis. In the optimization, the vector of design variables $\mathbf{s}$ is decomposed into two parts: cross-sectional dimensions $\mathbf{s}^1 = [b, h_i]$ and reinforcement $\mathbf{s}^2 = [(A_{s1})_i, (A_{s2})_i]$. For fixed b, h and fixed M, N, the minimal reinforcement can be computed. Hence, at each iteration step the vector $\mathbf{s}^1$ is found using the feasible direction method of nonlinear programming coupled with FEM, whereas the vector $\mathbf{s}^2$ is computed using the authors' procedure for minimization of $(A_{s1})_i + (A_{s2})_i$ separately for each cross-section.

The above formulated problem is continuum-type optimization problem, where the optimal design is found using nonlinear programming. In engineering practice discrete sets of reinforcements and dimensions b, h are used. The discrete solution can be approximated in two-step optimization:

(i) find continuum-type solution $(h, A_{s1}, A_{s2})_i$ for a fixed width of cross-sections b,

(ii) take h_i as next elements in a discrete set and find optimal reinforcement.

The solution obtained by this procedure satisfies all constraints, but it is only 'near to optimal'.

In practical design, even in case of built-up frames, the cross-sectional dimensions of beams and columns are unified within only few groups. Therefore, ussually, it is not difficult to improve the step (ii) by the way of systematic search of the descrete values of h_i in the vicinity of continuum-type solution (i). This procedure was used in the numerical examples presented in the paper.

4. Numerical examples

In the following examples the optimization was carried out for fixed width b of cross-sections and for the price coefficients: $c_c = 100$ PLZL/m^3 = 32 ECU/m^3, $c_s = 6500$ PLZL/m^3 = 2038 ECU/m^3, $c_f = 0$.

In order to study the influence of cost of formwork c_f on optimal design, in the example 4.2 of portal frame the optimal solutions for c_f = 2PLZL/m^2 were computed, too. The design strength of concrete $\bar{f}_{cd}$ = 11.5 MPa and the design strength of steel f_{yd} = 310 MPa were assumed in all numerical examples. These values correspond to concrete C20/25 and steel A-II [9, 10].

The reinforcement in the beams was computed for three sections: near the left support, in the mid of span and near the right support. Similarly, the reinforcement in the beams and columns of frames was computed for three sections of each rod.

4.1. Continuous beam

Consider a three-span continuous beam shown in Fig. 3. The optimal depth h and optimal reinforcement A_{s1}, A_{s2} in three sections of beams for various width b are presented in Table 1.

Table 1. Optimal dimensions h and reinforcement for the continuous beam

No	Assumed b [m]	Dimensional Design Variables	h [m]	Element No	Reinforcement ratio ρ compression [%]	Reinforcement ratio ρ tension [%]	Objective Function F [PLZL]
1	0.25	$h_1=h_2=h_3$	0.85	1	0.00-0.00-0.00	0.00-1.28-1.18	527.56
				2	0.00-0.00-0.00	1.18-0.56-1.18	
				3	0.00-0.00-0.00	1.18-1.28-0.00	
2		$h_1=h_3$	0.80	1	0.00-0.00-0.00	0.00-1.24-2.04	**505.96**
		h_2	0.55	2	2.23-0.51-2.23	4.45-2.74-4.45	
				3	0.00-0.00-0.00	2.04-1.24-0.00	
3	0.30	$h_1=h_2=h_3$	0.75	1	0.00-0.00-0.00	0.00-1.41-1.30	579.87
				2	0.00-0.00-0.00	1.30-0.61-1.30	
				3	0.00-0.00-0.00	1.30-1.41-0.00	
4		$h_1=h_3$	0.75	1	0.00-0.00-0.00	0.00-1.15-1.93	555.62
		h_2	0.50	2	2.44-0.63-2.44	4.60-2.85-4.60	
				3	0.00-0.00-0.00	1.93-1.15-0.00	
5	0.35	$h_1=h_2=h_3$	0.70	1	0.00-0.00-0.00	0.00-1.39-1.29	627.95
				2	0.00-0.00-0.00	1.29-0.60-1.29	
				3	0.00-0.00-0.00	1.29-1.39-0.00	
6		$h_1=h_3$	0.70	1	0.00-0.00-0.00	0.00-1.12-1.92	602.20
		h_2	0.45	2	2.30-0.82-2.30	4.53-3.04-4.53	
				3	0.00-0.00-0.00	1.92-1.12-0.00	

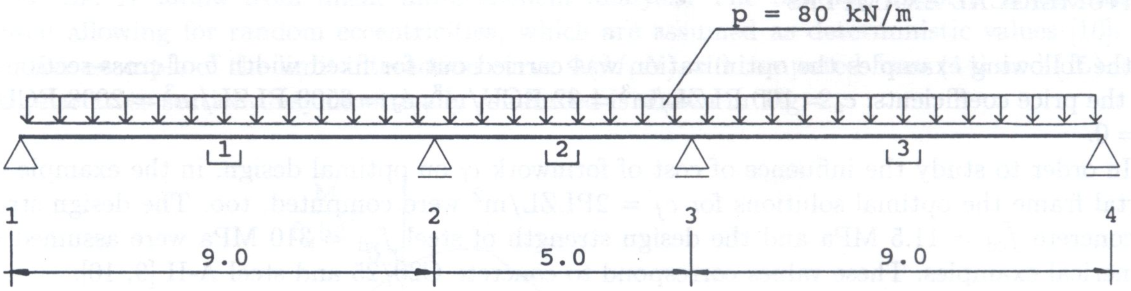

Fig. 3. Continuous beam

4.2. Portal frame

The frame and the loading are shown in Fig. 4. Table 2 presents 12 optimal designs computed for various prescribed values of width b and various number of independent variables h_i. For comparision the prices of formwork $c_f = 0$ and $c_f = 2$ PLZL/m^2 = 0.63 ECU/m^2 were considered. The optimal designs for the latter case are denoted in Table 2 by letter b and are printed in italics. In designs No 2, 3, 4, 5 the cost of formwork c_f did not influence the optimal values h_i and hence the optimal reinforcement remained the same, too, though the objective function was different. This was possible because the cost of formwork had only small influence on the continuum-type solution and the discrete solution remained the same for $c_f = 0$ and $c_f = 2$.

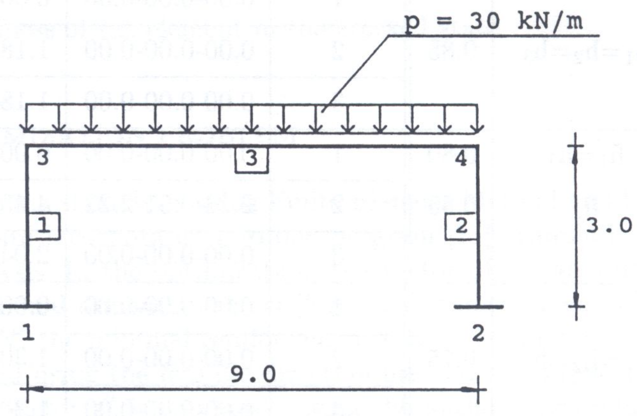

Fig. 4. Portal frame

In order to study the convergence, various initial designs were used as starting points in the optimization. The greatest reduction of the objective function was observed in the first iteration step and it depended on the initial design. The solution of optimum-type problem (i) was reached in 4 to 7 iteration steps, where each iteration step was connected with a call of FEM subprogram. It should be noted here that the number of iteration steps depends on the starting point and the required accuracy. The accuracy requirement is introduced by user supplied coefficients in stopping conditions in nonlinear programming [11].

The continuum-type optimal solution (step(i)) is a lower bound for a final discrete-type optimal solution (step(ii)). Table 3 illustrates the convergence by the way of 3 runs of optimization program.

Table 2. Optimal dimensions h and reinforcement for the portal frame (*in italics the optimal results obtained for* $c_f = 2$ *are given if they were different from results obtained for* $c_f = 0$)

No	Assumed b [m]	Dimensional Design Variables	h [m]	Element No	Reinforcement ratio compression [%]	Reinforcement ratio tension [%]	Objective Function F [PLZL]
				1	0.15	0.99	
					0.47	*1.30*	
1a		$h_1=h_2=h_3$	0.50	2	0.00-0.00-0.00	1.14-0.92-1.14	374.17
1b			*0.45*		*0.00-0.00-0.00*	*1.64-1.13-1.63*	*426.00*
				3	0.15	0.99	
	0.25				*0.47*	*1.30*	
		$h_1=h_3$	0.25	1	0.20	1.09	
2a			*0.25*				**314.10**
2b		h_2	0.65	2	0.00-0.00-0.00	0.15-1.02-0.15	***364.54***
			0.65	3	0.20	1.09	
				1	0.15	1.05	
3a		$h_1=h_2=h_3$	0.45	2	0.00-0.00-0.00	1.20-0.97-1.20	411.= 51
3b			*0.45*	3	0.15	1.05	*464.55*
	0.30	$h_1=h_3$	0.30	1	0.15	1.12	
4a			*0.30*				371.02
4b		h_2	0.60	2	0.00-0.00-0.00	0.25-0.87-0.25	*424.66*
			0.60	3	0.15	1.12	
				1	0.15	1.22	
5a		$h_1=h_2=h_3$	0.40	2	0.00-0.00-0.00	1.36-1.09-1.36	446.3= 5
5b			*0.40*	3	0.15	1.22	*499.39*
				1	0.15	1.01	
	0.35	$h_1=h_3$	0.35		*0.49*	*1.25*	
6a			*0.35*	2	0.00-0.00-0.00	0.39-0.74-0.39	428.24
6b		h_2	0.55		*0.00-0.00-0.00*	*0.62-0.78-0.62*	*484.60*
			0.50	3	0.15	1.01	
					0.49	*1.25*	

Table 3. Convergence of optimization of the portal frame (Example 2)

No	b [m]	Design No *Table 2*	Result	Initial Design $h_i[m]$ F [PLZL]	Continuum Step (i) Optimal Design $h_i[m]$ F [PLZL]	Continuum Step (i) Iteration number	Continuum Step (i) Analysis number	Discrete Step (ii) Optimal Design $h_i[m]$ F [PLZL]
			$h_1=h_3$	0.70	0.25			0.25
1	0.25	2a	h_2	0.70	0.6796	5	27	0.65
			F	404.58	312.95			314.10
			$h_1=h_3$	0.50	0.25			
2	0.25	2a	h_2	0.50	0.6809	6	33	
			F	375.32	313.09			
			$h_1=h_3$	0.70	0.30			0.30
3	0.30	4a	h_2	0.70	0.6340	4	21	0.60
			F	465.54	370.90			371.02

4.3. Three-storey frame

The frame and the loading are shown in Fig. 5. The optimal values of h_i for various widths b and for various number of independent variables h_i are presented in Table 4, whereas Table 5 shows the optimal reinforcement. For the sake of brevity in Table 5 the reinforcement A_{s1}, A_{s2} is presented only for one optimal design No 4 from Table 4, where four independent variables h_i were allowed for.

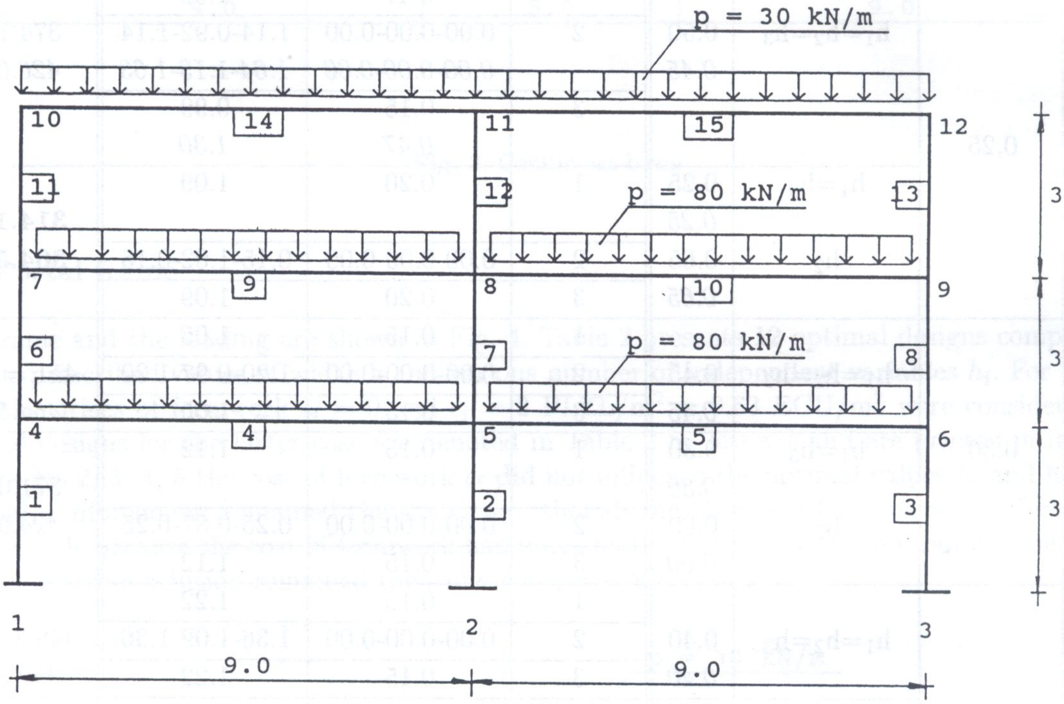

Fig. 5. Three-storey frame

Table 4. Optimal dimensions h for the 3-storey frame

No	Assumed b [m]	NDV	Dimensional Design Variables	h [m]	Objective Function [PLZL]
1	0.30	1	$h_1 \div h_{15}$	0.65	2626.6
2	0.30	2	$h_1=h_2=h_3=h_6=h_7=h_8=h_{11}=h_{=12}=h_{13}$	0.60	2604.9
			$h_4=h_5=h_9=h_{10}=h_{14}=h_{15}$	0.95	
3	0.30	3	$h_1=h_2=h_3=h_6=h_7=h_8=h_{11}=h_{=12}=h_{13}$	0.60	2603.0
			$h_4=h_5=h_9=h_{10}$	0.95	
			$h_{14}=h_{15}$	0.60	
4	0.30	4	$h_1=h_3=h_6=h_8$	0.60	**2601.3**
			$h_4=h_5=h_9=h_{10}$	0.95	
			$h_{11}=h_{13}=h_{14}=h_{15}$	0.55	
			$h_2=h_7=h_{12}$	0.60	

Table 5. Optimal reinforcement for four independent variables h

Assumed b [m]	Dimensional Design Variables	h [m]	Element No	Reinforcement ratio ρ compression [%]	Reinforcement ratio ρ tension [%]	Objective Function [PLZL]
0.30	$h_1=h_3=h_6=h_8$	0.60	1,3	0.15	0.15	2601.3
			6,8	0.36	0.35	
	$h_4=h_5=h_9=h_{10}$	0.95	4,5	0.00-0.00-0.00	0.45-0.38-0.78	
			9,10	0.00-0.00-0.00	0.46-0.38-0.76	
	$h_{11}=h_{13}=h_{14}=h_{15}$	0.55	11,13	0.52	0.51	
			14,15	0.00-0.00-0.00	0.87-0.42-0.87	
	$h_2=h_7=h_{12}$	0.60	2	0.15	0.15	
			7	0.15	0.15	
			12	0.15	0.15	

5. Concluding remarks

The formulation and solution of the problem of optimal design of reinforced concrete beams and frames were presented in the paper. Ultimate limit state (bearing capacity) and serviceability limit state (cracking and deflection) were allowed for in the constraints. Geometric constraints resulting from engineering practice were introduced, as well. Numerical examples were computed for a continuous beam and two frame structures. They proved practical applicability of the method. The proposed algorithm of optimization with the use of decomposition of design variables provided fast convergence. Reinforced concrete beams and frames are widely used in practice, therefore application of the theory of optimal design can facilitate the design process and diminish the cost of design and erection.

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