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Parameter sensitivity for large deformation inelastic problems

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The algorithm for parametric sensitivity assessment for both materially and geometrically nonlinear static problems is presented. Similarly to its geometrically linear version presented elsewhere, the sensitivity analysis is shown to reduce to a linear problem with the same operator matrix that has been used in just completed equilibrium iteration, which makes the computations very efficient.

Both total and updated Lagrangian approaches are analysed, including design differentiation of the configuration update transformation. Sensitivity with respect to constitutive parameters is discussed in detail. Possible extensions towards cross-sectional geometry or general shape parameters are pointed out.

1. INTRODUCTION

In the recent few years, sensitivity analysis of solid mechanics systems has become a subject of increasing interest of researchers and practitioners. Efforts of computational engineers used to be so far focused mainly on development of numerical methods and tools that would allow to obtain equilibrium solutions for various nonlinear mechanical problems with the best possible acuracy and at a reasonable computational cost. Recently, the progress in computer technology, development of computer oriented optimization techniques as well as several other factors have resulted in a growing need of having numerical equilibrium solutions accompanied by extended error analysis and perturbation studies. Numerical techniques of sensitivity analysis play the crucial role in solving computational engineering problems of this type.

There have been several papers published in recent years that deal with the parametric sensitivity analysis of elasto-plastic and elasto-viscoplastic structures [1–7, 9–12]. The most important features of the approach developed there can be summarized as follows:

- Even if the equilibrium problem is highly nonlinear, the sensitivity analysis at each time integration step is linear and no iterative routines are needed provided the consistent stiffness matrix is available,
- Possible design non-differentiability of inelastic response at some points along the deformation path is generally not a problem from the point of view of the algorithm stability or accuracy. However, discontinuity of the sensitivity results may make their reliability limited in the case of finite perturbation studies.

The aim of the paper is to extend the discussion initiated in the previous two papers by the authors, [3, 4], to the cases of large deformations including issues of configuration update and appropriate tensor transformations. Only sensitivity with respect to constitutive parameters will be addressed in detail. However, possibilities of straightforward extensions to the case of cross-section sizing as well as structure shape parameters will also be pointed out. The main conclusion is that, although the computations become much more tedious for this class of problems, the general characteristics of the sensitivity analysis itemized above remains valid.

2. Equilibrium problem

The geometrically nonlinear displacement problem of inelastic solid consists in space and time integration of the continuing equilibrium equation

$$\dot{\tau}_{ij,j} + \dot{\hat{f}}_i = 0\,, \qquad i,j = 1,2,3, \tag{1}$$

together with appropriate initial and boundary conditions and with the stress rate replaced by displacement rates (velocities) by means of additional equations discussed below. The symbol τ_{ij} should be understood as components of the first Piola-Kirchhoff stress tensor related to a chosen reference configuration C^{r} while $\hat{f}_i$ denotes external volume forces.

To express the stress in terms of the displacement field the following additional equations are necessary:

- geometric equations that relate strain ε_{ij} to deformation gradient F_{ij} and displacement u_i

$$\varepsilon_{ij} = \varepsilon^0_{ij} + \tfrac{1}{2}(F_{ki}F_{kj} - \delta_{ij})\,, \tag{2}$$

$$F_{ij} = \delta_{ij} + u_{i,j} \tag{3}$$

  where ε^0_{ij} describes strain at the reference configuration (its value is of minor importance here since strain enters the formulation in the form of time derivative only), and u_i is defined as displacement field with respect to the reference (and not initial) configuration,

- constitutive relations between second Piola-Kirchhoff stress rate $\dot{\sigma}_{ij}$ and strain rate $\dot{\varepsilon}_{ij}$, assumed in a general form valid for both history-dependent and history-independent materials

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}(\sigma_{kl}, p_\gamma, \dot{\varepsilon}_{kl}) \tag{4}$$

  where p_γ denotes a set of scalar as well as tensorial internal variables typical of particular form of history-dependent behaviour,

- constitutive relations describing evolution of the parameters p_γ

$$\dot{p}_\delta = \dot{p}_\delta(\sigma_{kl}, p_\gamma, \dot{\varepsilon}_{kl})\,, \tag{5}$$

- geometric equations that relate the first to second Piola-Kirchhoff stress measures

$$\tau_{ij} = F_{ik}\sigma_{kj}\,. \tag{6}$$

It is assumed that both the constitutive equations (4), (5) depend on $\dot{\varepsilon}_{ij}$ only linearly.

Employing the displacement-based finite element method [13] in which the displacement field $u_i({}^{\mathrm{r}}x_k, t)$ is approximated by a linear combination of nodal displacements $q_\alpha(t)$ and shape functions $\psi_{i\alpha}({}^{\mathrm{r}}x_k)$,

$$u_i({}^{\mathrm{r}}x_k, t) = \psi_{i\alpha}({}^{\mathrm{r}}x_k)\, q_\alpha(t)\,, \qquad (\alpha \text{ runs over the set of nodal parameters}) \tag{7}$$

the resulting semi-discrete system of equations becomes

$$\int_{\Omega^{\mathrm{r}}} \tilde{B}_{ij\alpha}\, \dot{\sigma}_{ij}\, \mathrm{d}\Omega^{\mathrm{r}} = \dot{Q}_\alpha \tag{8}$$

in which Q_α is the vector of nodal loads

$$Q_\alpha = \int_{\Omega^{\mathrm{r}}} \hat{f}_i\, \psi_{i\alpha}\, \mathrm{d}\Omega^{\mathrm{r}} + \int_{\partial\Omega^{\mathrm{r}}_\sigma} \hat{t}_i\, \psi_{i\alpha}\, \mathrm{d}(\partial\Omega^{\mathrm{r}}) \tag{9}$$

(with $\hat{f}_i$ and $\hat{t}_i$ being known volume and surface loads, respectively) and $\dot{\sigma}_{ij}$ is expressed in terms of the nodal displacement rate vector $\dot{q}_\alpha$ via the unchanged constitutive equation (4) and a new discretized geometric equation

$$\varepsilon_{ij} = \varepsilon^0_{ij} + B_{ij\alpha}(q_\beta)\, q_\alpha \,, \tag{10}$$

$$B_{ij\alpha} = \overline{B}_{ij\alpha} + q_\beta\, \overline{\overline{B}}_{ij\alpha\beta} \,, \tag{11}$$

$$\overline{B}_{ij\alpha} = \tfrac{1}{2}\left(\psi_{i\alpha,j} + \psi_{j\alpha,i}\right) \,, \tag{12}$$

$$\overline{\overline{B}}_{ij\alpha\beta} = \tfrac{1}{2}\psi_{k\beta,i}\,\psi_{k\alpha,j} \,, \tag{13}$$

while

$$\tilde{B}_{ij\alpha} = \frac{\mathrm{d}\varepsilon_{ij}}{\mathrm{d}q_\alpha} = \overline{B}_{ij\alpha} + 2\overline{\overline{B}}_{ij\alpha\beta}\, q_\beta \,. \tag{14}$$

Remark: The above space-discretized formulation applies only to a limited class of finite element analyses, namely, to the analysis with the use of continuum elements. A similar formulation valid for structural elements (shells, beams) may be obtained by replacing the stresses and strains in the above equations with generalized cross-sectional forces and strains typical of the structure considered, and 3-D space integration with integration in 2-D or 1-D characteristic subspaces of such structural model. In such a formulation, the constitutive equation in the discretized model have to be also replaced with a generalized constitutive equation in which cross-section sizing parameters (like thickness or inertia moments) appear as additional constitutive parameters. Since the cross-section forces and strains usually cannot be written in a compact tensor notation, the equilibrium problem formulation appears more complex and elaborate; it remains in full analogy to the above one, though. Thus, in order to make our discussion clear and readable, we will not explicitly address such cases in this paper — an extension of the formulation is possible and requires some more tedious geometric derivations only. □

To numerically integrate in time the above equations together with the constitutive equations (4), (5), an integration scheme has to be employed. In it, time derivatives are replaced by finite time increments and a step-by-step procedure is employed to yield the 'end-of-the-step' (say, at time $t + \Delta t$) solution using the 'beginning-of-the-step' (at time t) data derived from the previous step calculations. Equation (8) takes after such an integration the following form

$$\int_{\Omega^{\mathrm{r}}} \tilde{B}_{ij\alpha}\left({}^t\sigma_{ij} + \Delta\sigma_{ij}\right) \mathrm{d}\Omega^{\mathrm{r}} = {}^{t+\Delta t}Q_\alpha \tag{15}$$

where ${}^t\sigma_{ij}$ is considered known from the previous time step while $\Delta\sigma_{ij}$ is expressed as a function of the unknown displacement vector Δq_α by the incremental form of the constitutive equation and the geometric equation, the latter written in a form useful when using an iterative scheme as

$$\delta\Delta\varepsilon_{ij} = \tilde{B}_{ij\alpha}\, \delta\Delta q_\alpha \,. \tag{16}$$

There are two crucial issues of the algorithm in the case of inelastic and geometrically nonlinear problems:

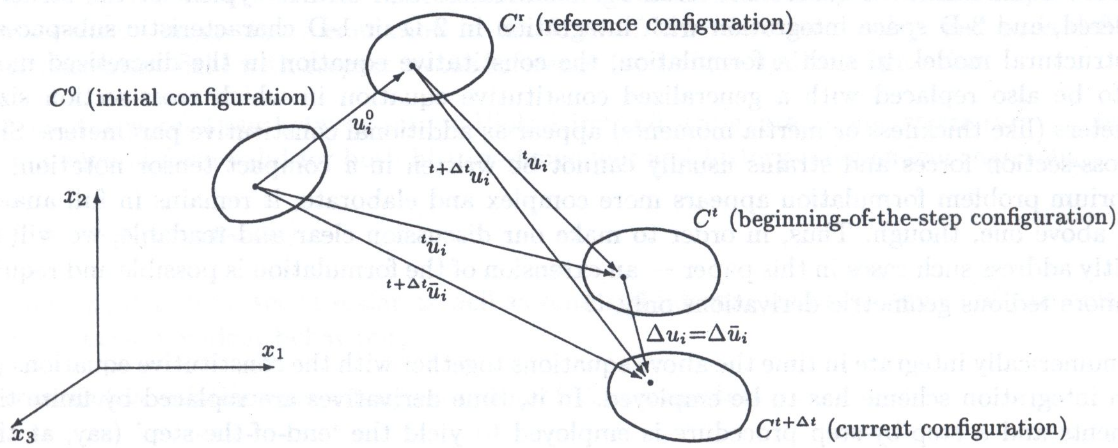
- The way in which the incremental stress is computed from the incremental strain, i.e. the time integration scheme for the constitutive law. This issue will not be discussed in much detail in this paper. We only mention that currently the most commonly used implicit time integration scheme for elasto-plastic as well as elasto-viscoplastic problems is the so-called "return mapping" algorithm known in the case of the Huber-Mises yield function as the "radial return" method. It was put forward by Simo and Taylor [8] for integration of small-strain elasto-plastic equations. It can be formally extended to large-strain analysis as well, although more advanced schemes referring to deformation gradient decomposition seem nowadays to be more often employed for this class of problems.

Given the integration scheme, the rate-type constitutive equations can be replaced by their incremental counterparts as

$$\Delta\sigma_{ij} = \Delta\sigma_{ij}({}^t\sigma_{kl}, {}^tp_\gamma, \Delta\varepsilon_{kl}) , \tag{17}$$

$$\Delta p_\delta = \Delta p_\delta({}^t\sigma_{kl}, {}^tp_\gamma, \Delta\varepsilon_{kl}) . \tag{18}$$

- The way in which the reference configuration $C^{\rm r}$ is chosen and possibly updated at subsequent steps of the analysis. There are two approaches to this issue commonly employed in computational practice: (i) total Lagrangian, in which the reference configuration is kept constant during the whole analysis, and (ii) updated Lagrangian, in which the initial configuration at each time step (i.e. that at the time instant t) becomes the reference configuration for this step, cf. Fig. 1. The total Lagrangian approach is conceptually simple and convenient from the point of view of computational costs. On the other hand, in the updated Lagrangian approach we deal with much more realistic tensorial process measures which is why this approach is regarded to be more 'natural' and reliable.



Total Lagrangian: $C^{\rm r} = C^0, \quad u_i^0 = 0$; ${}^tu_i = {}^t\bar{u}_i , \quad u_i = \bar{u}_i$

Updated Lagrangian: $C^{\rm r} = C^t, \quad u_i^0 = {}^t\bar{u}_i$; ${}^tu_i = 0, \quad u_i = \Delta u_i$

Fig. 1

The nonlinear equilibrium equation (15) expressed in terms of unknown nodal displacement increments can be iteratively solved with the use of the Newton-Raphson method. In it, subsequent corrections $\delta q_\alpha^{(\omega)}$ to the solution Δu_α are computed from the equation

$${}^{t+\Delta t}K_{\alpha\beta}^{(\omega-1)}\ \delta q_\beta^{(\omega)} = {}^{t+\Delta t}Q_\alpha - {}^{t+\Delta t}R_\alpha^{(\omega-1)}, \quad \omega = 1, 2, \ldots , \tag{19}$$

where

$${}^{t+\Delta t}R_\alpha^{(\omega-1)} = \int_{\Omega^{\rm r}} \tilde{B}_{ij\alpha}^{(\omega-1)}\ {}^{t+\Delta t}\sigma_{ij}^{(\omega-1)}\ {\rm d}\Omega^{\rm r} , \tag{20}$$

$$\tilde{B}_{ij\alpha}^{(\omega-1)} = \tilde{B}_{ij\alpha}({}^{t+\Delta t}q_\beta^{(\omega-1)}) , \tag{21}$$

$${}^{t+\Delta t}\sigma_{ij}^{(\omega-1)} = {}^t\sigma_{ij} + \Delta\sigma_{ij}({}^{t+\Delta t}q_\alpha^{(\omega-1)}) , \tag{22}$$

$${}^{t+\Delta t}q_\alpha^{(\omega-1)} = {}^tq_\alpha + \Delta q_\alpha^{(\omega-1)} , \tag{23}$$

and

$$K_{\alpha\beta} = \frac{\mathrm{d}R_\alpha}{\mathrm{d}\Delta q_\beta} = \int_{\Omega^{\mathrm{r}}} \left(2\overline{\overline{B}}_{ij\alpha\beta}\,\sigma_{ij} + \tilde{B}_{ij\alpha}\,C_{ijkl}\,\tilde{B}_{kl\beta}\right) \mathrm{d}\Omega^{\mathrm{r}} \,, \tag{24}$$

$$C_{ijkl} = \frac{\mathrm{d}\Delta\sigma_{ij}}{\mathrm{d}\Delta\varepsilon_{kl}} \,, \tag{25}$$

are the consistent (with the time integration scheme) tangent stiffness matrices at the global (structural) and local (material) level, respectively. As we shall see soon, even if the use of other, not necessarily tangent matrices may lead to correct equilibrium solution (with only some loss in the convergence rate), the use of the consistent stiffness matrices has crucial importance in view of the very nature of sensitivity analysis and not only its computational cost.

Once Eq. (15) has been solved with respect to Δq_α, nodal displacements as well as stress and internal parameters at material points (i.e., rather, at Gauss integration points) have to be updated so that they can be used in the computations at the next time step:

$$^{t+\Delta t}q_\alpha = {}^{t}q_\alpha + \Delta q_\alpha \,, \tag{26}$$

$$^{t+\Delta t}\sigma_{ij} = {}^{t}\sigma_{ij} + \Delta\sigma_{ij} \,, \tag{27}$$

$$^{t+\Delta t}p_\gamma = {}^{t}p_\gamma + \Delta p_\gamma \,. \tag{28}$$

The Cauchy stress measure in the current configuration $C^{t+\Delta t}$ can be obtained from the following formula

$$\varsigma_{ij} = \frac{1}{J}\,F_{ik}\,F_{jl}\,\sigma_{kl} \tag{29}$$

where J stands for the deformation gradient determinant, $J = \det F_{ij}$.

Finally, if the updated Lagrangian approach is employed, the reference configuration coordinates have to be updated and stress tensors (as well as possibly other tensorial quantities included in the set of internal variables p_γ) have to be redefined in the new configuration before they can be used in the constitutive equation for the next time step:

$$q^{\mathrm{r}}_\alpha := q^{\mathrm{r}}_\alpha + \Delta q_\alpha \,, \tag{30}$$

$$q_\alpha := 0 \,, \tag{31}$$

$$\sigma_{ij} := \varsigma_{ij} \,, \tag{32}$$

$$^{\mathrm{r}}x_i := {}^{\mathrm{r}}x_i + \Delta u_i \,. \tag{33}$$

Here, we have used the fact that the second Piola-Kirchhoff stress tensor on the new reference configuration (i.e. the one at $t + \Delta t$) is equal to the Cauchy stress tensor.

Remark: The procedure of configuration update described above may look very simplistic to a careful reader. Indeed, additional tedious derivations are necessary to redefine in the new configuration all the fourth-rank tensors of the material coefficients in the constitutive equations, for instance. This issue will not be addressed in this paper; it is assumed that the constitutive equation in each new configuration is known. In fact, the changes in constitutive coefficients due to the configuration update are frequently neglected in the computational practice. □

3. Sensitivity problem

3.1. Basic idea of analysis

Consider a performance function Φ dependent on some state variables and the constitutive parameters h in the form[1]

$$\Phi = \Phi(\sigma_{ij}, p_\gamma, \bar{q}_\alpha; h) \tag{34}$$

with $\bar{q}_\alpha$ denoting total nodal displacement with respect to the initial configuration,

$$\bar{q}_\alpha = q^0_\alpha + q_\alpha \, . \tag{35}$$

The function Φ is treated here as given which means that its value can be directly computed once values of the arguments are known. Clearly, the values of σ_{ij} , p_γ and $\bar{q}_\alpha$ should be considered to depend on h as well

$$\Phi = \Phi(\sigma_{ij}(h), p_\gamma(h), \bar{q}_\alpha(h); h). \tag{36}$$

Our goal in the design sensitivity analysis is to compute the total derivative of Φ with respect to h, i.e.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}h} = \frac{\mathrm{d}\Phi}{\mathrm{d}\sigma_{ij}}\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}h} + \frac{\mathrm{d}\Phi}{\mathrm{d}p_\gamma}\frac{\mathrm{d}p_\gamma}{\mathrm{d}h} + \frac{\mathrm{d}\Phi}{\mathrm{d}\bar{q}_\alpha}\frac{\mathrm{d}\bar{q}_\alpha}{\mathrm{d}h} + \frac{\partial\Phi}{\partial h} \, . \tag{37}$$

There are many ways to determine the design derivatives on the right hand side of the above equation. Surely, all the ways consist in design-differentiation of the equilibrium problem and its solution with respect to design derivative of the displacement field which can be then substituted into geometric and constitutive equations to yield $\mathrm{d}\Phi/\mathrm{d}h$. In one of the possible approaches the continuum problem (1) is first design-differentiated and then discretized in time and space (not necessarily in the same way as the equilibrium problem), solved with respect to $\mathrm{d}u_i/\mathrm{d}h$, and then integrated and finally substituted into (37). In another approach, it is the FEM discretized formulation that is design-differentiated, solved with respect to $\mathrm{d}\Delta q_\alpha/\mathrm{d}h$ at each time step, and then substituted into discretized geometric and constitutive equations and finally into (37). Both the approaches are presented schematically in Fig. 2 and denoted by (A) and (B), respectively. Obviously, other approaches lying somewhere between the above two ones can be considered, too.

In this paper, only the approach (B) is considered. It means that we shall be seeking the design sensitivity of the numerical solution rather than of the continuum one. Thus, we will determine the value of $\mathrm{d}\Phi/\mathrm{d}h$ at the end of each time step assuming that the parameters that Φ depends on are expressed as

$$\sigma_{ij}(h) = {}^t\sigma_{ij}(h) + \Delta\sigma_{ij}\left({}^t\sigma_{kl}(h), {}^tp_\gamma(h), \Delta\varepsilon_{kl}({}^tq_\alpha(h), \Delta q_\alpha(h)); h\right) \tag{38}$$

$$p_\delta(h) = {}^tp_\delta(h) + \Delta p_\delta\left({}^t\sigma_{kl}(h), {}^tp_\gamma(h), \Delta\varepsilon_{kl}({}^tq_\alpha(h), \Delta q_\alpha(h)); h\right) \tag{39}$$

$$\bar{q}_\alpha(h) = q^0_\alpha(h) + {}^tq_\alpha(h) + \Delta q_\alpha(h) \tag{40}$$

3.2. Remarks on notation

In order to make our notation simpler, we shall be further using design variations rather than design derivatives. The design variation is defined as

$$\delta(\cdot) = \frac{\mathrm{d}(\cdot)}{\mathrm{d}h}\delta h \tag{41}$$

[1]For the sake of better transparency of the derivations to follow, we shall restrict ourselves to the sensitivity analysis with just one performance function and one design parameter. Our discussion may easily be extended to more general and realistic problems with larger numbers of both the quantities.

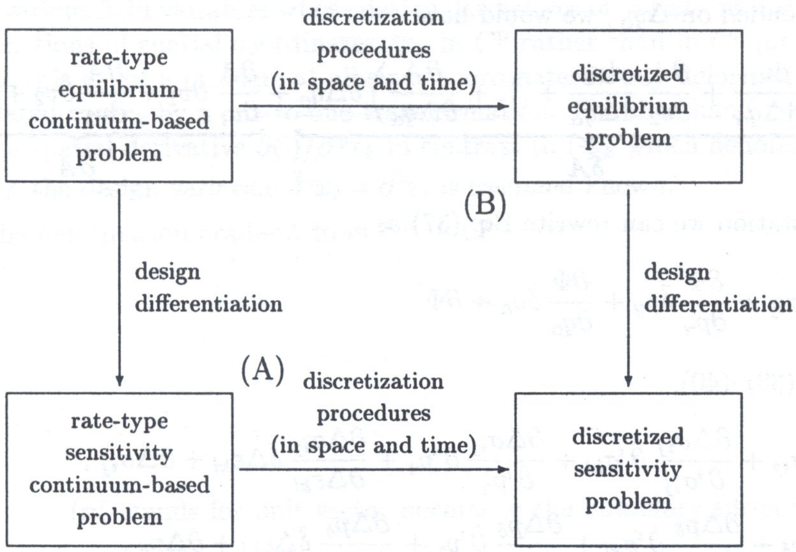


Fig. 2

and, as it is linear in δh, both notions can be used interchangeably.

Accounting for the nature of the rate-dependent constitutive equation and the incremental approach employed for the equilibrium analysis a few different design variations will be distinguished:

- implicit design variation — it is the part of the total design variation that is related to variation of the incremental equilibrium solution Δq_α,

$$\tilde{\delta}(\cdot) = \frac{\mathrm{d}(\cdot)}{\mathrm{d}\Delta q_\alpha}\,\bar{\delta}\Delta q_\alpha \tag{42}$$

(obviously, $\bar{\delta}\Delta q_\alpha \equiv \tilde{\delta}\Delta q_\alpha$),

- effectively explicit design variation — it is the remaining part of the total design variation related to design variations of all other variables that the considered quantity depends on

$$\hat{\partial}(\cdot) = \bar{\delta}(\cdot) - \tilde{\delta}(\cdot) = \left.\frac{\mathrm{d}(\cdot)}{\mathrm{d}h}\right|_{\Delta q_\alpha \neq \Delta q_\alpha(h)} \delta h\,, \tag{43}$$

- the latter variation should not be confused with the following truly explicit design variation defined as

$$\partial(\cdot) = \frac{\partial(\cdot)}{\partial h}\,\delta h \tag{44}$$

for quantities that are explicit functions of the design parameters.

To make the notation more clear still, if a quantity A is defined as a function

$$A = A(v_1(h), v_2(h), \ldots, \Delta q_\alpha; h)$$

where $v_1, v_2, \ldots$ are some quantities that do not depend on Δq_α, then

$$\bar{\delta}A = \underbrace{\frac{\partial A}{\partial \Delta q_\alpha}\,\bar{\delta}\Delta q_\alpha}_{\tilde{\delta}A} + \underbrace{\frac{\partial A}{\partial v_1}\,\hat{\partial}v_1 + \frac{\partial A}{\partial v_2}\,\hat{\partial}v_2 + \ldots + \partial A}_{\hat{\partial}A}\,.$$

If $v_1, v_2, \ldots$ depended on Δq_α, we would have

$$\bar{\delta}A = \underbrace{\left(\frac{\partial A}{\partial v_1}\frac{\mathrm{d}v_1}{\mathrm{d}\Delta q_\alpha} + \frac{\partial A}{\partial v_2}\frac{\mathrm{d}v_2}{\mathrm{d}\Delta q_\alpha} + \ldots + \frac{\partial A}{\partial \Delta q_\alpha}\right)\bar{\delta}\Delta q_\alpha}_{\tilde{\delta}A} + \underbrace{\frac{\partial A}{\partial v_1}\hat{\partial}v_1 + \frac{\partial A}{\partial v_2}\hat{\partial}v_2 + \ldots + \partial A}_{\hat{\partial}A} .$$

With this notation we can rewrite Eq. (37) as

$$\bar{\delta}\Phi = \frac{\partial \Phi}{\partial \sigma_{ij}}\bar{\delta}\sigma_{ij} + \frac{\partial \Phi}{\partial p_\gamma}\bar{\delta}p_\gamma + \frac{\partial \Phi}{\partial q_\alpha}\bar{\delta}\bar{q}_\alpha + \partial\Phi \tag{45}$$

where, cf. Eqs. (38)–(40),

$$\bar{\delta}\sigma_{ij} = \hat{\partial}\,{}^t\sigma_{ij} + \frac{\partial \Delta\sigma_{ij}}{\partial\,{}^t\sigma_{ij}}\hat{\partial}\,{}^t\sigma_{kl} + \frac{\partial \Delta\sigma_{ij}}{\partial\,{}^t p_\gamma}\hat{\partial}\,{}^t p_\gamma + \frac{\partial \Delta\sigma_{ij}}{\partial \Delta\varepsilon_{kl}}\bar{\delta}\Delta\varepsilon_{kl} + \partial\Delta\sigma_{ij} , \tag{46}$$

$$\bar{\delta}p_\delta = \hat{\partial}\,{}^t p_\delta + \frac{\partial \Delta p_\delta}{\partial\,{}^t\sigma_{ij}}\hat{\partial}\,{}^t\sigma_{kl} + \frac{\partial \Delta p_\delta}{\partial\,{}^t p_\gamma}\hat{\partial}\,{}^t p_\gamma + \frac{\partial \Delta p_\delta}{\partial \Delta\varepsilon_{kl}}\bar{\delta}\Delta\varepsilon_{kl} + \partial\Delta p_\delta , \tag{47}$$

$$\bar{\delta}\Delta\varepsilon_{ij} = \frac{\partial \Delta\varepsilon_{kl}}{\partial\,{}^t q_\alpha}\hat{\partial}\,{}^t q_\alpha + \frac{\partial \Delta\varepsilon_{kl}}{\partial \Delta q_\alpha}\tilde{\delta}\Delta q_\alpha , \tag{48}$$

$$\bar{\delta}\bar{q}_\alpha = \hat{\partial}q^0_\alpha + \hat{\partial}\,{}^t q_\alpha + \tilde{\delta}\Delta q_\alpha . \tag{49}$$

3.3. Design-independent configuration

At this point we have to make a comment on the important issue of reference configuration in the sensitivity analysis. First, we observe that quantities like stresses, strains, deformation gradients etc., depend on the design parameter h not only via constitutive or geometric equations. The dependence is much more complex, since the definitions of the above quantities involve derivatives with respect to coordinates ${}^r x$ in the reference configuration C^r which is generally design-dependent as well. More precisely, in the updated Lagrangian description C^r is the current deformed configuration and thus it is always design-dependent, while in the total Lagrangian (as well as at the first step in the updated Lagrangian) description C^r is the initial configuration which is usually design-independent (although it may depend on design in case of shape sensitivity problems). Design differentiation of quantities like stresses or strains may therefore be difficult unless they are expressed in a new, design-independent configuration.

Let us introduce a fictitious reference (or parent) configuration C^p and assume it to be design-independent, cf. Fig. 3. The most natural way to define such a configuration in finite element practice is to identify it with the parent configurations of particular isoparametric elements used.

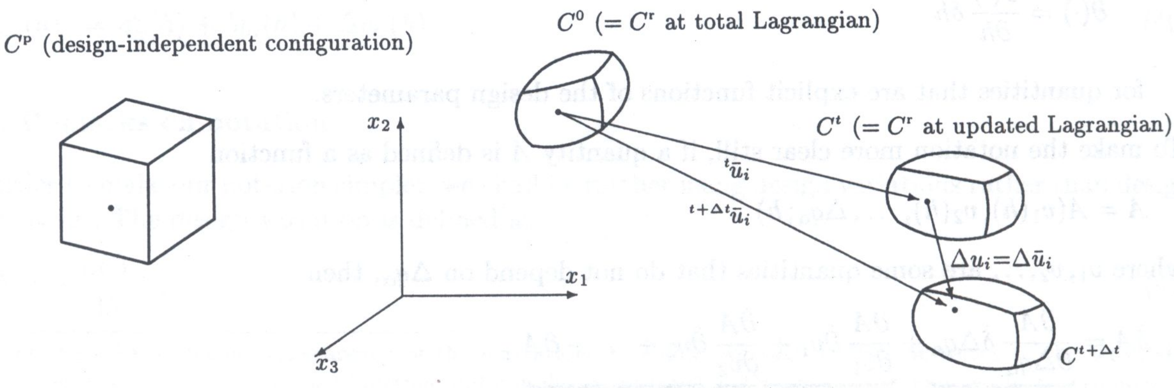


Fig. 3

In all further derivations field variables whose design derivatives or variations are considered will be assumed to be functions of spatial coordinates ${}^{\mathrm{p}}x_k$ in C^{p} rather than in C^{r} (or any other configuration). Formally, this is not a problem at all since coordinates in any deformed or undeformed configuration are related to ${}^{\mathrm{p}}x_k$ by a one-to-one transformation. The shorthand notation $(\cdot)_{;k}$ will be used to denote the spatial derivative $\partial(\cdot)/\partial\,{}^{\mathrm{p}}x_k$ in contrast to $(\cdot)_{,k}$ which denotes the derivative $\partial(\cdot)/\partial\,{}^{\mathrm{r}}x_k$. Obviously, the design variation $\bar{\delta}\,{}^{\mathrm{r}}x_i = \hat{\partial}\,{}^{\mathrm{r}}x_i$ is assumed known.

Defining $\overset{*}{F}_{ij}$ as the deformation gradient from C^{p} to C^{r}

$$\overset{*}{F}_{ij} = {}^{\mathrm{r}}x_{i;j} \tag{50}$$

and denoting

$$\overset{*}{J} = \det \overset{*}{F}_{ij}\,, \tag{51}$$

$$\overset{*}{J}_\partial = \overset{*}{J}\left|n^{\mathrm{p}}_j \overset{*}{F}{}^{-1}_{ji}\right| \qquad (n^{\mathrm{p}}_j \text{ stands for unit vector normal to the boundary surface in } C^{\mathrm{p}}), \tag{52}$$

we can write down the following formulae to be used in further discussion of the sensitivity problems (note that shape functions $\psi_{i\alpha}$ are expressed in terms of ${}^{\mathrm{p}}x_k$ and thus design independent):

$$(\cdot)_{,i} = (\cdot)_{;j}\, \overset{*}{F}{}^{-1}_{ji}\,, \tag{53}$$

$$\bar{\delta}\overset{*}{F}{}^{-1}_{ji} = -\overset{*}{F}{}^{-1}_{jk}\,\bar{\delta}\overset{*}{F}_{kl}\,\overset{*}{F}{}^{-1}_{li} = -\overset{*}{F}{}^{-1}_{jk}\,(\bar{\delta}\,{}^{\mathrm{r}}x_k)_{;l}\,\overset{*}{F}{}^{-1}_{li} = -\overset{*}{F}{}^{-1}_{jk}\,(\bar{\delta}\,{}^{\mathrm{r}}x_k)_{,i}\,, \tag{54}$$

$$\bar{\delta}(\cdot)_{,i} = \bar{\delta}(\cdot)_{;j}\,\overset{*}{F}{}^{-1}_{ji} + (\cdot)_{;j}\,\bar{\delta}\overset{*}{F}{}^{-1}_{ji} = (\bar{\delta}(\cdot))_{,i} - (\cdot)_{,k}\,(\bar{\delta}\,{}^{\mathrm{r}}x_k)_{,i}\,, \tag{55}$$

$$\bar{\delta}\,{}^{t+\Delta t}F_{ij} = \bar{\delta}\,{}^{t+\Delta t}x_{i,j} = (\bar{\delta}\,{}^{t+\Delta t}x_i)_{,j} - x_{i,k}\,(\hat{\partial}\,{}^{\mathrm{r}}x_k)_{,j}\,, \tag{56}$$

$$\bar{\delta}J = J\,F^{-1}_{ji}\,\bar{\delta}F_{ij}\,, \tag{57}$$

$$\bar{\delta}\bar{B}_{ij\alpha} = \hat{\partial}\bar{B}_{ij\alpha} = \tfrac{1}{2}\left(\bar{\delta}\psi_{i\alpha,j} + \bar{\delta}\psi_{j\alpha,i}\right) = -\tfrac{1}{2}\left[\psi_{i\alpha,k}\,(\hat{\partial}\,{}^{\mathrm{r}}x_k)_{,j} + \psi_{j\alpha,k}\,(\hat{\partial}\,{}^{\mathrm{r}}x_k)_{,i}\right], \tag{58}$$

$$\bar{\delta}\bar{\bar{B}}_{ij\alpha\beta} = \hat{\partial}\bar{\bar{B}}_{ij\alpha\beta} = \tfrac{1}{2}\hat{\partial}\left(\psi_{k\alpha,i}\,\psi_{k\alpha,j}\right) = -\tfrac{1}{2}\left[\psi_{k\alpha,l}\,\psi_{k\alpha,j}\,(\hat{\partial}\,{}^{\mathrm{r}}x_l)_{,i} + \psi_{k\alpha,i}\,\psi_{k\alpha,l}\,(\hat{\partial}\,{}^{\mathrm{r}}x_l)_{,j}\right], \tag{59}$$

$$\bar{\delta}\tilde{B}_{ij\alpha} = \underbrace{2\bar{\bar{B}}_{ij\alpha\beta}\,\tilde{\delta}\Delta q_\beta}_{\tilde{\delta}\tilde{B}_{ij\alpha}} + \underbrace{\hat{\partial}\bar{B}_{ij\alpha} + 2\hat{\partial}\bar{\bar{B}}_{ij\alpha\beta}({}^{t}q_\beta + \Delta q_\beta) + 2\bar{\bar{B}}_{ij\alpha\beta}\hat{\partial}\,{}^{t}q_\beta}_{\hat{\partial}\tilde{B}_{ij\alpha}}\,, \tag{60}$$

$$\bar{\delta}\Delta\varepsilon_{ij} = \underbrace{\tilde{B}_{ij\alpha}\,\tilde{\delta}\Delta q_\alpha}_{\tilde{\delta}\Delta\varepsilon_{ij}} + \underbrace{\left(\bar{\delta}\bar{B}_{ij\alpha} + 2\hat{\partial}\bar{\bar{B}}_{ij\alpha\beta}\,{}^{t}q_\beta + 2\bar{\bar{B}}_{ij\alpha\beta}\,\hat{\partial}\,{}^{t}q_\beta + \hat{\partial}\bar{\bar{B}}_{ij\alpha\beta}\Delta q_\beta\right)\Delta q_\alpha}_{\hat{\partial}\Delta\varepsilon_{ij}}\,. \tag{61}$$

Note that vector ${}^{t}q_\alpha$ is equal to zero when the updated Lagrangian approach is employed.

Further, we have

$$\int_{\Omega^{\mathrm{r}}} (\cdot)\,\mathrm{d}\Omega^{\mathrm{r}} = \int_{\Omega^{\mathrm{p}}} (\cdot)\,\overset{*}{J}\,\mathrm{d}\Omega^{\mathrm{p}}\,, \tag{62}$$

$$\bar{\delta}\left[\int_{\Omega^{\mathrm{r}}} (\cdot)\,\mathrm{d}\Omega^{\mathrm{r}}\right] = \int_{\Omega^{\mathrm{p}}}\left[\bar{\delta}(\cdot)\,\overset{*}{J} + (\cdot)\,\bar{\delta}\overset{*}{J}\right]\mathrm{d}\Omega^{\mathrm{p}} = \int_{\Omega^{\mathrm{r}}}\left[\bar{\delta}(\cdot) + (\cdot)\,\frac{\bar{\delta}\overset{*}{J}}{\overset{*}{J}}\right]\mathrm{d}\Omega^{\mathrm{r}}\,, \tag{63}$$

$$\int_{\partial\Omega^{\mathrm{r}}} (\cdot)\,\mathrm{d}(\partial\Omega^{\mathrm{r}}) = \int_{\partial\Omega^{\mathrm{p}}} (\cdot)\,\overset{*}{J}_\partial\,\mathrm{d}(\partial\Omega^{\mathrm{p}})\,, \tag{64}$$

$$\bar{\delta}\left[\int_{\partial\Omega^{\mathrm{r}}} (\cdot)\,\mathrm{d}(\partial\Omega^{\mathrm{r}})\right] = \int_{\partial\Omega^{\mathrm{p}}}\left[\bar{\delta}(\cdot)\,\overset{*}{J}_\partial + (\cdot)\,\bar{\delta}\overset{*}{J}_\partial\right]\mathrm{d}(\partial\Omega^{\mathrm{p}}) = \int_{\partial\Omega^{\mathrm{r}}}\left[\bar{\delta}(\cdot) + (\cdot)\,\frac{\bar{\delta}\overset{*}{J}_\partial}{\overset{*}{J}_\partial}\right]\mathrm{d}(\partial\Omega^{\mathrm{r}})\,, \tag{65}$$

with

$$\bar{\delta} \overset{*}{J} = \hat{\partial} \overset{*}{J} = \overset{*}{J}\, \bar{\delta} \overset{*}{F}_{ij} \overset{*}{F}{}^{-1}_{ji} = \overset{*}{J}\, (\bar{\delta}\, {}^{\mathrm{r}}x_i)_{,i}\,, \tag{66}$$

$$\bar{\delta} \overset{*}{J}_{\partial} = \hat{\partial} \overset{*}{J}_{\partial} = \overset{*}{J}_{\partial}\, \bar{\delta} \overset{*}{F}_{ij} \overset{*}{F}{}^{-1}_{jk} \left(\delta_{ik} - \frac{n^{\mathrm{p}}_l \overset{*}{F}{}^{-1}_{lk}\, n^{\mathrm{p}}_m \overset{*}{F}{}^{-1}_{mi}}{n^{\mathrm{p}}_l \overset{*}{F}{}^{-1}_{ln}\, n^{\mathrm{p}}_m \overset{*}{F}{}^{-1}_{mn}} \right) = \overset{*}{J}_{\partial}\, (\bar{\delta}\, {}^{\mathrm{r}}x_i)_{,j} \left(\delta_{ij} - n^{\mathrm{r}}_i n^{\mathrm{r}}_j \right). \tag{67}$$

3.4. Solution method

Let us rewrite Eqs. (45)–(49) in the following form

$$\bar{\delta}\Phi = \hat{\partial}\Phi + \tilde{\delta}\Phi \tag{68}$$

where

$$\hat{\partial}\Phi = \frac{\partial\Phi}{\partial\sigma_{ij}}\, \hat{\partial}\sigma_{ij} + \frac{\partial\Phi}{\partial p_\gamma}\, \hat{\partial}p_\gamma + \frac{\partial\Phi}{\partial\bar{q}_\alpha} \left(\hat{\partial} q^0_\alpha + \hat{\partial}\, {}^t q_\alpha \right) + \partial\Phi\,, \tag{69}$$

$$\tilde{\delta}\Phi = \frac{\partial\Phi}{\partial\sigma_{ij}}\, \tilde{\delta}\sigma_{ij} + \frac{\partial\Phi}{\partial p_\gamma}\, \tilde{\delta}p_\gamma + \frac{\partial\Phi}{\partial\bar{q}_\alpha}\, \tilde{\delta}\Delta q_\alpha = \frac{\mathrm{d}\Phi}{\mathrm{d}\Delta q_\alpha}\, \tilde{\delta}\Delta q_\alpha\,, \tag{70}$$

$$\hat{\partial}\sigma_{ij} = \hat{\partial}\, {}^t\sigma_{ij} + \frac{\partial\Delta\sigma_{ij}}{\partial\, {}^t\sigma_{ij}}\, \hat{\partial}\, {}^t\sigma_{kl} + \frac{\partial\Delta\sigma_{ij}}{\partial\, {}^t p_\gamma}\, \hat{\partial}\, {}^t p_\gamma + \frac{\partial\Delta\sigma_{ij}}{\partial\Delta\varepsilon_{kl}}\, \hat{\partial}\Delta\varepsilon_{kl}\,, \tag{71}$$

$$\hat{\partial}p_\delta = \hat{\partial}\, {}^t p_\delta + \frac{\partial\Delta p_\delta}{\partial\, {}^t\sigma_{ij}}\, \hat{\partial}\, {}^t\sigma_{kl} + \frac{\partial\Delta p_\delta}{\partial\, {}^t p_\gamma}\, \hat{\partial}\, {}^t p_\gamma + \frac{\partial\Delta p_\delta}{\partial\Delta\varepsilon_{kl}}\, \hat{\partial}\Delta\varepsilon_{kl}\,, \tag{72}$$

$$\tilde{\delta}\sigma_{ij} = \frac{\partial\Delta\sigma_{ij}}{\partial\Delta\varepsilon_{kl}}\, \tilde{\delta}\Delta\varepsilon_{kl} = C_{ijkl}\, \tilde{\delta}\Delta\varepsilon_{kl}\,, \tag{73}$$

$$\tilde{\delta}p_\delta = \frac{\partial\Delta p_\delta}{\partial\Delta\varepsilon_{kl}}\, \tilde{\delta}\Delta\varepsilon_{kl}\,, \tag{74}$$

with $\hat{\partial}\Delta\varepsilon_{kl}$ and $\tilde{\delta}\Delta\varepsilon_{kl}$ defined in Eq. (61). The variation $\hat{\partial}\Phi$ contains terms that can easily be determined once the equilibrium solution for this time step is known and sensitivities of stresses, internal parameters and nodal displacements have been computed and stored at the previous time step. The terms contributing to $\tilde{\delta}\Phi$ contain the vector $\tilde{\delta}\Delta q_\alpha$ which is the basic unknown in the sensitivity problem.

In order to determine it, the incremental finite element equilibrium equation (15) has to be differentiated with respect to design. By making use of Eq. (9), let us rewrite it as

$$\int_{\Omega^{\mathrm{r}}} \tilde{B}_{ij\alpha}\, \sigma_{kj}\, \mathrm{d}\Omega^{\mathrm{r}} = \int_{\Omega^{\mathrm{r}}} \hat{f}_i\, \psi_{i\alpha}\, \mathrm{d}\Omega^{\mathrm{r}} + \int_{\partial\Omega^{\mathrm{r}}_\sigma} \hat{t}_i\, \psi_{i\alpha}\, \mathrm{d}(\partial\Omega^{\mathrm{r}}) \tag{75}$$

or, shorter, as

$$R_\alpha = Q_\alpha\,. \tag{76}$$

All the quantities in the above equation refer to the end of the time step, i.e. to the time instant $t + \Delta t$. Taking design variation of this equation and applying auxiliary formulae derived in Section 3.3 we finally obtain

$$\tilde{\delta}R_\alpha = \hat{\partial}Q_\alpha - \hat{\partial}R_\alpha \tag{77}$$

where

$$\tilde{\delta}R_\alpha = \int_{\Omega^{\mathrm{r}}} \left(\tilde{\delta}\tilde{B}_{ij\alpha}\, \sigma_{kj} + \tilde{B}_{ij\alpha}\, \tilde{\delta}\sigma_{kj} \right) \mathrm{d}\Omega^{\mathrm{r}} = K_{\alpha\beta}\, \tilde{\delta}\Delta q_\beta\,, \tag{78}$$

$$\hat{\partial} R_\alpha = \int_{\Omega^r} \left(\hat{\partial}\tilde{B}_{ij\alpha}\,\sigma_{kj} + \tilde{B}_{ij\alpha}\,\hat{\partial}\sigma_{kj} + \tilde{B}_{ij\alpha}\,\sigma_{kj}\,\frac{\hat{\partial}\overset{*}{J}}{\overset{*}{J}} \right) \mathrm{d}\Omega^r \,, \tag{79}$$

$$\hat{\partial} Q_\alpha = \int_{\Omega^r} \left(\hat{\partial}\hat{f}_i + \hat{f}_i\,\frac{\hat{\partial}\overset{*}{J}}{\overset{*}{J}} \right) \psi_{i\alpha}\,\mathrm{d}\Omega^r + \int_{\partial\Omega^r_\sigma} \left(\hat{\partial}\hat{t}_i + \hat{t}_i\,\frac{\hat{\partial}\overset{*}{J}_\partial}{\overset{*}{J}_\partial} \right) \psi_{i\alpha}\,\mathrm{d}(\partial\Omega^r) \,, \tag{80}$$

with $\hat{\partial}\tilde{B}_{ij\alpha}$ and $\bar{\delta}\tilde{B}_{ij\alpha}$ given in Eq. (60) while $\hat{\partial}\sigma_{ij}$, $\bar{\delta}\sigma_{ij}$, $\hat{\partial}\overset{*}{J}$ and $\hat{\partial}\overset{*}{J}_\partial$ given in Eqs. (71), (73), (66) and (67), respectively.

Note that similarly as in the geometrically linear problems, [3], the sensitivity problem (77)–(80) consists in the solution of a set of linear equations with the coefficient matrix equal to the consistent stiffness matrix of the structure analysed. If this matrix is used in the Newton-Raphson iterations for the equilibrium analysis, its last decomposed form will be available at the stage of the sensitivity analysis which makes the latter a very cheap add-on to a typical FEM equilibrium analysis code.

Once $\bar{\delta}\Delta q_\alpha$ is determined, the performance variation can be computed from Eq. (45). Also, by substituting $\bar{\delta}\Delta q_\alpha$ in the expressions for $\bar{\delta}\sigma_{ij}$ and $\bar{\delta}p_\gamma$, cf. Eqs. (73), (74), we can update sensitivities of the stress and internal parameters at the integration points so that they can be used for the computations at the next time step that begins at $t + \Delta t$.

The design variation of the current Cauchy stress can be obtained by taking the design variation of Eq. (29) which in view of Eq. (57) yields

$$\bar{\delta}\varsigma_{ij} = \frac{1}{J}\left(-F^{-1}_{nm}\,\bar{\delta}F_{mn}\,F_{ik}\,F_{jl} + \bar{\delta}F_{ik}\,F_{jl} + F_{ik}\,\bar{\delta}F_{jl} \right)\sigma_{kl} + \frac{1}{J}\,F_{ik}\,F_{jl}\,\bar{\delta}\sigma_{kl} \tag{81}$$

with $\bar{\delta}F_{ij}$ given in Eq. (56).

The last procedure that has to be performed at the end of each time step is the redefinition of the coordinates and updated stress sensitivites in the new physical reference configuration $C^r = C^{t+\Delta t}$ in the case of the updated Lagrangian approach. By taking design variations of Eqs. (32), (33) we obtain

$$\bar{\delta}q^r_\alpha := \bar{\delta}q^r_\alpha + \bar{\delta}\Delta q_\alpha \,, \tag{82}$$

$$\bar{\delta}q_\alpha := 0 \,, \tag{83}$$

$$\bar{\delta}\sigma_{ij} := \bar{\delta}\varsigma_{ij} \,, \tag{84}$$

$$\bar{\delta}\,{}^r x_i := \bar{\delta}\,{}^r x_i + \bar{\delta}\Delta u_i \,. \tag{85}$$

Remark: The computational algorithm of sensitivity analysis described above is referred to in the literature as the direct differentiation method (DDM). There exists an alternative approach called the adjoint system method (ASM). Its main idea can be explained by taking reference to Eqs. (68), (70), (77) and (78):

$$\bar{\delta}\Phi = \hat{\partial}\Phi + \underbrace{\frac{\mathrm{d}\Phi}{\mathrm{d}\Delta q_\alpha}\,K^{-1}_{\alpha\beta}}_{\lambda_\beta}\!\!\!\!\!\!\!\!\overbrace{\phantom{K^{-1}_{\alpha\beta}}\,\hat{\partial}(Q_\beta - R_\beta)}^{\bar{\delta}\Delta q_\alpha} \,. \tag{86}$$

Vector λ_α defined above is usually called the 'adjoint vector'. ASM consists in solving a system of equations with the coefficient matrix $K_{\alpha\beta}$ and the right-hand side $\mathrm{d}\Phi/\mathrm{d}\Delta q_\alpha$ and then substituting it to the above equation. ASM is similar to DDM in that in both the methods vectors $\hat{\partial}(Q_\alpha - R_\alpha)$ and $\mathrm{d}\Phi/\mathrm{d}\Delta q_\alpha$ have to be assembled and a system of equations with the consistent stiffness matrix has to be solved. The differences can be summarized as follows:

- in ASM, neither the nodal displacement sensitivity nor the sensitivities of stresses or internal parameters are ever explicitly computed (unless particular components of these quantities are chosen as the response Φ)
- in ASM, the number of necessary back-substitutions is proportional to the number of performance functionals Φ while in DDM — to the number of design variables h, which makes ASM competitive in problems with large number of design variables and small number of response functionals, and DDM in the opposite case.

In the case of history-dependent analysis when the sensitivities of usually a large number of variables have to be monitored and updated through the whole history of deformations, the use of ASM would mean in practice that each stress component and internal parameter at each integration point, as well as each nodal displacement, have to be defined as additional performance functionals. This makes ASM practically useless for such problems since, as it has just been mentioned, its efficiency is poor if the number of response functionals exceeds the number of design variables.

4. Computational example

The theoretical derivations will now be illustrated with a simple computational example. Its goal is to display the whole track of computations necessary to obtain the sensitivity of a possibly simplest structure with both material and geometric nonlinearities.

4.1. Constitutive equation

The well known constitutive model of Huber–Mises elasto-plasticity with isotropic hardening will be adopted. Details of its rate-type formulation as well as 'return mapping' time integration scheme can be found in [8] while the complete differentiation of the time-discretized constitutive equations with respect to material parameters was derived in [3]. Here, the discretized equations will be only briefly recalled. The set of internal constitutive parameters consists in this case of just one variable — the equivalent plastic strain $\bar{e}^{\mathrm{p}}$. The material constants are: Young modulus E, Poisson ratio ν, yield limit σ_{y} and hardening parameter ζ. Further material constants in the form of shear modulus G and bulk mudulus K, are expressed as

$$G = \frac{E}{2(1+\nu)}\,, \qquad K = \frac{E}{3(1-2\nu)}\,.$$

The incremental stress is expressed as

$$\Delta\sigma_{ij} = C^{\mathrm{e}}_{ijkl}(\Delta\varepsilon_{kl} - \Delta\varepsilon^{\mathrm{p}}_{kl}) \tag{87}$$

where

$$C^{\mathrm{e}}_{ijkl} = K\,\delta_{ij}\delta_{kl} + 2G\left(I_{ijkl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}\right), \qquad I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \tag{88}$$

$$\Delta\varepsilon^{\mathrm{p}}_{ij} = \sqrt{\tfrac{3}{2}}\,\Delta\bar{e}^{\mathrm{p}}\, n_{ij}\,, \tag{89}$$

and n_{ij} is the norm of the trial stress deviator

$$n_{ij} = \frac{s^{\mathrm{tr}}_{ij}}{\|s^{\mathrm{tr}}_{kl}\|}\,, \qquad s^{\mathrm{tr}}_{ij} = {}^{t}s_{ij} + 2G\Delta\varepsilon_{ij}\,. \tag{90}$$

The value of $\Delta\bar{e}^{\mathrm{p}}$ depends on the yield criterion defined by the function

$$f({}^{t}\sigma_{ij}, {}^{t}\bar{e}^{\mathrm{p}}, \Delta\varepsilon_{ij}) = \bar{\sigma}^{\mathrm{tr}} - (\sigma_{\mathrm{y}} + \zeta\,{}^{t}\bar{e}^{\mathrm{p}})\,, \qquad \bar{\sigma}^{\mathrm{tr}} = \sqrt{\tfrac{3}{2}}\,\|s^{\mathrm{tr}}_{kl}\|\,. \tag{91}$$

If $f \le 0$ then $\Delta\bar{e}^{\mathrm{p}} = 0$ while otherwise

$$\Delta\bar{e}^{\mathrm{p}} = \frac{\gamma f}{3G}\,, \qquad \gamma = \frac{1}{1+\frac{\zeta}{3G}} \tag{92}$$

which satisfies the flow rule

$${}^{t+\Delta t}\bar{\sigma} - (\sigma_{\mathrm{y}} + \zeta\, {}^{t+\Delta t}\bar{e}^{\mathrm{p}}) = 0\,. \tag{93}$$

It is easy to prove that the incremental constitutive equations defined above for the stress (87) and equivalent plastic strain (92) after substitution of all remaining equations written above may be expressed in the general form (17)–(18).

The consistent stiffness matrix for the above incremental formulation has the form, [3, 8]

$$C_{ijkl} = K\,\delta_{ij}\delta_{kl} + 2G(1-\gamma)\left(I_{ijkl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}\right) - 2G(\gamma-\vartheta)\, n_{ij}\, n_{kl} \tag{94}$$

with

$$\vartheta = \frac{3G\,\Delta\bar{e}^{\mathrm{p}}}{\bar{\sigma}^{\mathrm{tr}}}\,. \tag{95}$$

Assuming that all material constants in the model may depend on the design parameter h (and this dependence is explicitly known) we can differentiate the above equations with respect to design and obtain the following relations for the total stress and internal parameter design variations, cf. (46)–(47),

$$\bar{\delta}\sigma_{ij} = \hat{\partial}\,{}^{t}\sigma_{ij} + \frac{\partial\Delta\sigma_{ij}}{\partial\,{}^{t}\sigma_{ij}}\hat{\partial}\,{}^{t}\sigma_{kl} + \frac{\partial\Delta\sigma_{ij}}{\partial\,{}^{t}\bar{e}^{\mathrm{p}}}\hat{\partial}\,{}^{t}\bar{e}^{\mathrm{p}} + C_{ijkl}\,\bar{\delta}\Delta\varepsilon_{kl} + \partial\Delta\sigma_{ij}\,, \tag{96}$$

$$\bar{\delta}\bar{e}^{\mathrm{p}} = \hat{\partial}\,{}^{t}\bar{e}^{\mathrm{p}} + \frac{\partial\Delta\bar{e}^{\mathrm{p}}}{\partial\,{}^{t}\sigma_{ij}}\hat{\partial}\,{}^{t}\sigma_{kl} + \frac{\partial\Delta\bar{e}^{\mathrm{p}}}{\partial\,{}^{t}\bar{e}^{\mathrm{p}}}\hat{\partial}\,{}^{t}\bar{e}^{\mathrm{p}} + \frac{\partial\Delta\bar{e}^{\mathrm{p}}}{\partial\Delta\varepsilon_{kl}}\bar{\delta}\Delta\varepsilon_{kl} + \partial\Delta\bar{e}^{\mathrm{p}}\,, \tag{97}$$

where

$$\frac{\partial\Delta\sigma_{ij}}{\partial\,{}^{t}\sigma_{ij}} = -\vartheta\left(I_{ijkl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}\right) - (\gamma-\vartheta)\, n_{ij}\, n_{kl}\,, \tag{98}$$

$$\frac{\partial\Delta\sigma_{ij}}{\partial\,{}^{t}\bar{e}^{\mathrm{p}}} = \sqrt{\tfrac{2}{3}}\,\gamma\,\zeta\, n_{ij}\,, \tag{99}$$

$$\begin{aligned}\partial\Delta\sigma_{ij} &= \partial K\,\delta_{ij}\Delta\varepsilon_{kk} + \sqrt{\tfrac{2}{3}}\,\gamma\,(\partial\sigma_{\mathrm{y}} + \bar{e}^{\mathrm{p}}\,\partial\zeta)\, n_{ij} \\ &\quad + \left[(1-\vartheta)\left(\Delta\varepsilon_{ij} - \tfrac{1}{3}\,\delta_{ij}\Delta\varepsilon_{kk}\right) - (1-\gamma)\,\Delta\varepsilon^{\mathrm{p}}_{ij} - (\gamma-\vartheta)\, n_{ij}\, n_{kl}\,\Delta\varepsilon_{kl}\right]\partial(2G)\,,\end{aligned} \tag{100}$$

$$\frac{\partial\Delta\bar{e}^{\mathrm{p}}}{\partial\,{}^{t}\sigma_{ij}} = \frac{\gamma}{3G}\sqrt{\tfrac{3}{2}}\, n_{ij}\,, \tag{101}$$

$$\frac{\partial\Delta\bar{e}^{\mathrm{p}}}{\partial\,{}^{t}\bar{e}^{\mathrm{p}}} = -\frac{\gamma\zeta}{3G}\,, \tag{102}$$

$$\partial\Delta\bar{e}^{\mathrm{p}} = -\frac{\gamma}{3G}\left[\partial\sigma_{\mathrm{y}} + \bar{e}^{\mathrm{p}}\,\partial\zeta + \tfrac{3}{2}\,\Delta\bar{e}^{\mathrm{p}}\,\partial(2G)\right]\,, \tag{103}$$

$$\frac{\partial\Delta\bar{e}^{\mathrm{p}}}{\partial\Delta\varepsilon_{ij}} = \sqrt{\tfrac{2}{3}}\,\gamma\, n_{ij}\,, \tag{104}$$

$$\partial K = \frac{1}{3(1-2\nu)}(\partial E + 6K\partial\nu)\,, \qquad \partial(2G) = \frac{1}{1+\nu}(\partial E - 2G\partial\nu)\,. \tag{105}$$

The equations above do not include the configuration update transformations, i.e. the total stress at $t+\Delta t$ as well as its design variation computed from these equations are expressed in the reference configuration at t.

In the computational example below the following values of material constants are assumed: $E = 1000$, $\nu = 0.25$ (i.e. $2G = 800$, $K = 666.67$), $\sigma_{\mathrm{y}} = 1$, $\zeta = 150$. Only one design parameter $h = E$ is considered.

4.2. Geometry and load

A uniform cube with unit dimensions loaded in three principal directions by three independent stretching forces is considered (Fig. 4a). This means that no shear stresses or strains occur in the model. Unknown parameters q_α are the three relative displacements of the cube walls due to elongation in each principal direction.

A monotonic loading history is defined in Fig. 4b. Load values at the time instant $t = 1$ are $\hat{t}_1 = 8$ and $\hat{t}_2 = \hat{t}_3 = -2$.

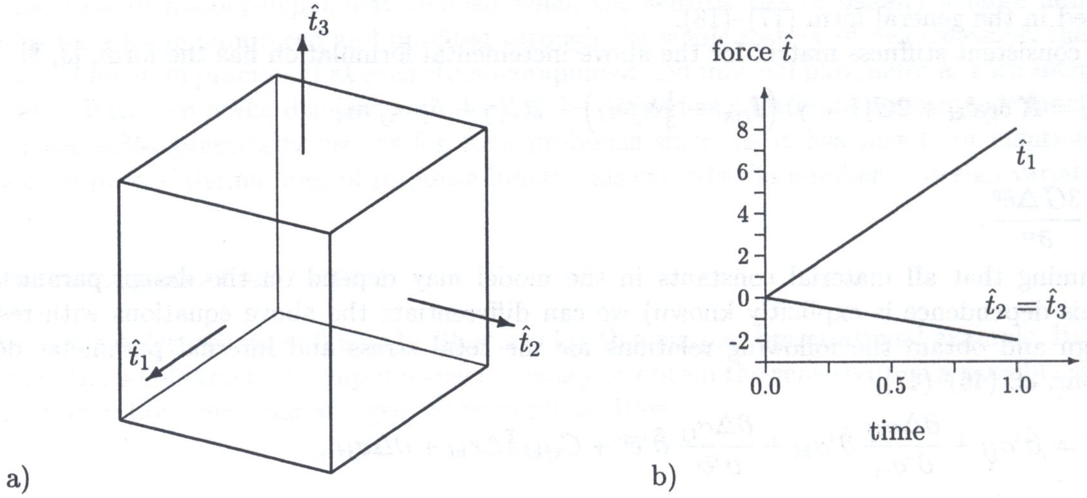


Fig. 4

Two loading steps, each of the length $\Delta t = 0.5$, are considered starting from the time instant $t = 0$. Updated Lagrangian approach is employed. Values of all crucial scalar variables and components of tensors that appear in the theoretical formulation of the equilibrium and sensitivity problem are displayed for each time step (in sensitivity computations, values of design derivatives rather than their variations are given). Details of equilibrium iterations are skipped and only the final solution is presented for each time step.

We assume that the design-independent configuration C^{p} is the initial, undeformed configuration at $t = 0$.

4.3. Results

In order to make it possible to display the values of tensor components a simplified version of the commonly used finite-element vector/matrix notation is employed below. We take advantage of the fact that non-diagonal entries in the Cartesian representation for stress, strain and deformation gradient tensors are zero in this example. The detailed description of the notational conventions goes as follows:

- index notation employed in the previous part of the paper is replaced by absolute (bold-face) notation for each tensor quantity.
- stress, strain and deformation gradient tensors are represented by 3×1 vectors of their diagonal component values,
- fourth-rank tensors in constitutive equations are replaced by 3×3 matrices; in particular, tensors I_{ijkl} and $\delta_{ij}\delta_{kl}$ are represented by $\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$ and $\begin{bmatrix} 1&1&1\\1&1&1\\1&1&1 \end{bmatrix}$, respectively,

- nodal displacement vector q_α is represented by a 3×1 vector of relative displacements of the cube walls; accordingly, R_α and Q_α are 3×1 vectors while stiffness matrix $K_{\alpha\beta}$ is a 3×3 matrix,
- geometric operators $\overline{B}_{ij\alpha}$, $\tilde{B}_{ij\alpha}$ as well as the product $\overline{\overline{B}}_{ij\alpha\beta}\,\Delta q_\beta$ are represented by 3×3 matrices (containing only components for $i = j$, all the remaining ones are zero) while $\overline{\overline{B}}_{ij\alpha\beta}$ itself becomes in such a notation an object of the third rank and will not be displayed explicitly.

First step, $t = 0$, $\Delta t = 0.5$

Incremental equilibrium problem

Initial values:

$$
\begin{aligned}
{}^t\mathbf{q} &= \{0,\,0,\,0\}\,,\\
{}^t\boldsymbol{\sigma} &= \{0,\,0,\,0\}\,,\\
{}^t\bar{e}^{\mathrm{p}} &= 0\,.
\end{aligned}
$$

Results:

$$
\begin{aligned}
\mathbf{Q} &= \{4.000000,\,-1.000000,\,-1.000000\}\,, \qquad \text{cf. Eq. (9)},\\
\Delta\mathbf{q} &= \{0.02993522,\,-0.01483832,\,-0.01483832\}\,,\\
\mathbf{F} &= \{1.0299352,\,0.9851617,\,0.9851617\}\,,\\
J &= 0.9995970\,,\\
\overline{\mathbf{B}} &= \mathbf{I}\,,\\
\overline{\overline{\mathbf{B}}}\,\Delta\mathbf{q} &= \begin{bmatrix} 0.02993522 & 0 & 0\\ 0 & -0.01483832 & 0\\ 0 & 0 & -0.01483832 \end{bmatrix},\\
\Delta\boldsymbol{\varepsilon} &= \{0.03038328,\,-0.01472824,\,-0.01472824\}\,,\\
\boldsymbol{\sigma}^{\mathrm{tr}} &= \{24.05947,\,-12.02974,\,-12.02974\}\,,\\
\bar{\sigma} &= 36.08921\,,\\
\Delta\bar{e}^{\mathrm{p}} &= 0.02599201\,,\\
\bar{e}^{\mathrm{p}} &= 0.02599201\,,\\
\sigma_{\mathrm{y}} + \zeta\bar{e}^{\mathrm{p}} &= 4.898801\,,\\
1-\vartheta &= 0.1357414\,,\\
\mathbf{n} &= \{0.816497,\,-0.408248,\,-0.408248\}\,,\\
\boldsymbol{\sigma} &= \{3.883739,\,-1.015062,\,-1.015062\}\,,\\
\tilde{\mathbf{B}} &= \begin{bmatrix} 1.0299352 & 0 & 0\\ 0 & 0.9851617 & 0\\ 0 & 0 & 0.9851617 \end{bmatrix},\\
\Omega^t &= 1.000000\,,\\
\mathbf{R} &= \{4.000000,\,-1.000000,\,-1.000000\}\,, \qquad \text{cf. Eq. (20)},\\
\mathbf{C} &= \begin{bmatrix} 725.9259 & 637.0370 & 637.0370\\ 637.0370 & 735.7781 & 627.1849\\ 637.0370 & 627.1849 & 735.7781 \end{bmatrix}, \qquad \text{cf. Eq. (25)},
\end{aligned}
$$

$$\mathbf{K} = \begin{bmatrix} 773.9217 & 646.3714 & 646.3714 \\ 646.3714 & 713.0896 & 608.7103 \\ 646.3714 & 608.7103 & 713.0896 \end{bmatrix}, \qquad \text{cf. Eq. (25)},$$

$$\varsigma = \{4.121402, -0.9855589, -0.9855589\}, \qquad \text{cf. Eq. (29)}.$$

Sensitivity analysis ($h = E$)

Initial values:

$$\frac{\mathrm{d}\,{}^t\mathbf{q}}{\mathrm{d}h} = \{0, 0, 0\},$$

$$\frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}}{\mathrm{d}h} = \{0, 0, 0\},$$

$$\frac{\mathrm{d}\,{}^t\bar{e}^{\mathrm{p}}}{\mathrm{d}h} = 0,$$

$$\frac{\partial K}{\partial h} = 0.666667, \qquad \frac{\partial(2G)}{\partial h} = 0.8, \qquad \frac{\partial \sigma_{\mathrm{y}}}{\partial h} + {}^t\bar{e}^{\mathrm{p}}\frac{\partial \zeta}{\partial h} = 0,$$

$$\psi_{11,1} = \psi_{22,2} = \psi_{33,3} = 1, \qquad \text{remaining } \psi_{i\alpha,j} = 0.$$

Right-hand side vector components:

$$\frac{\partial \mathbf{Q}}{\partial h} = \{0, 0, 0\},$$

$$\frac{\partial \Delta\boldsymbol{\sigma}}{\partial h} = 10^{-3} \times \{0.9807460, 0.4364349, 0.4364349\}, \qquad \text{cf. Eq. (100)},$$

$$\frac{\partial \Delta\boldsymbol{\sigma}}{\partial\,{}^t\boldsymbol{\sigma}}\frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}}{\mathrm{d}h} = \{0, 0, 0\}, \qquad \text{cf. Eq. (98)},$$

$$\frac{\partial \Delta\boldsymbol{\sigma}}{\partial\,{}^t\bar{e}^{\mathrm{p}}}\frac{\mathrm{d}\,{}^t\bar{e}^{\mathrm{p}}}{\mathrm{d}h} = \{0, 0, 0\}, \qquad \text{cf. Eq. (99)},$$

$$\frac{\mathrm{d}\overline{\mathbf{B}}}{\mathrm{d}h} = \begin{bmatrix} \psi_{11,1}^2 \frac{\mathrm{d}\,{}^tq_1}{\mathrm{d}h} & 0 & 0 \\ 0 & \psi_{22,2}^2 \frac{\mathrm{d}\,{}^tq_2}{\mathrm{d}h} & 0 \\ 0 & 0 & \psi_{33,3}^2 \frac{\mathrm{d}\,{}^tq_3}{\mathrm{d}h} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{cf. Eq. (58)},$$

$$\frac{\mathrm{d}\overline{\overline{\mathbf{B}}}}{\mathrm{d}h}\Delta\mathbf{q} = \begin{bmatrix} \psi_{11,1}^3 \frac{\mathrm{d}\,{}^tq_1}{\mathrm{d}h}\Delta q_1 & 0 & 0 \\ 0 & \psi_{22,2}^3 \frac{\mathrm{d}\,{}^tq_2}{\mathrm{d}h}\Delta q_2 & 0 \\ 0 & 0 & \psi_{33,3}^3 \frac{\mathrm{d}\,{}^tq_3}{\mathrm{d}h}\Delta q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

cf. Eq. (59),

$$\left.\frac{\mathrm{d}\Delta\boldsymbol{\varepsilon}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = \{0, 0, 0\}, \qquad \text{cf. Eq. (61)},$$

$$\left.\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = \frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}}{\mathrm{d}h} + \frac{\partial\Delta\boldsymbol{\sigma}}{\partial h} + \frac{\partial\Delta\boldsymbol{\sigma}}{\partial\,{}^t\boldsymbol{\sigma}}\frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}}{\mathrm{d}h} + \frac{\partial\Delta\boldsymbol{\sigma}}{\partial\,{}^t\bar{e}^{\mathrm{p}}}\frac{\mathrm{d}\,{}^t\bar{e}^{\mathrm{p}}}{\mathrm{d}h} + \mathbf{C}\left.\frac{\mathrm{d}\Delta\boldsymbol{\varepsilon}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)}$$

$$= 10^{-3} \times \{0.9807460, 0.4364349, 0.4364349\}, \qquad \text{cf. Eqs. (96), (61)},$$

$$\frac{\mathrm{d}\tilde{\mathbf{B}}}{\mathrm{d}h} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{cf. Eq. (60)},$$

$$\frac{1}{\overset{*}{J}}\frac{\mathrm{d}\overset{*}{J}}{\mathrm{d}h} = 0,$$

$$\left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = 10^{-3} \times \{1.0101048,\ 0.4299590,\ 0.4299590\}, \qquad \text{cf. Eq. (79).}$$

Results:

$$\frac{\mathrm{d}\Delta\mathbf{q}}{\mathrm{d}h} = \mathbf{K}^{-1} \left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = 10^{-6}\times\{-4.159140,\ 1.708572,\ 1.708572\},$$

(incremental sensitivity solution),

$$\frac{\mathrm{d}\Delta\boldsymbol{\varepsilon}}{\mathrm{d}h} = 10^{-6} \times \{-4.283644,\ 1.683219,\ 1.683219\}, \qquad \text{cf. Eq. (61),}$$

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}h} = 10^{-5} \times \{1.568352,\ 0.1760428,\ 0.1760428\}, \qquad \text{cf. Eq. (96),}$$

$$\frac{\mathrm{d}\bar{e}^{\mathrm{p}}}{\mathrm{d}h} = 9.282065 \times 10^{-8}, \qquad \text{cf. Eq. (97),}$$

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}h} = 10^{-6} \times \{-4.159140,\ 1.708572,\ 1.708572\}, \qquad \text{cf. Eq. (56),}$$

$$\frac{\mathrm{d}\varsigma}{\mathrm{d}h} = 10^{-5} \times \{-1.429554,\ -0.2270676,\ -0.2270676\}, \qquad \text{cf. Eq. (81).}$$

Second step, $t = 0.5,\ \Delta t = 1.0$

Incremental equilibrium problem

Initial values:

$$^{t}\mathbf{q} = \{0.02993522,\ -0.01483832,\ -0.01483832\},$$
$$^{t}\boldsymbol{\sigma} = \{4.121402,\ -0.9855589,\ -0.9855589\},$$
$$^{t}\bar{e}^{\mathrm{p}} = 0.02599201.$$

Results:

$$\mathbf{Q} = \{8.000000,\ -2.000000,\ -2.000000\}, \qquad \text{cf. Eq. (9),}$$
$$\Delta\mathbf{q} = \{0.03846379,\ -0.01847462,\ -0.01847462\},$$
$$\mathbf{F} = \{1.0373458,\ 0.9812471,\ 0.9812471\},$$
$$J = 0.9988042,$$
$$\overline{\mathbf{B}} = \{0.9709349,\ 1.0150618,\ 1.0150618\},$$
$$\overline{\overline{\mathbf{B}}}\,\Delta\mathbf{q} = \begin{bmatrix} 0.03626038 & 0 & 0 \\ 0 & -0.01903533 & 0 \\ 0 & 0 & -0.01903533 \end{bmatrix},$$
$$\Delta\boldsymbol{\varepsilon} = \{0.03804319,\ -0.01857705,\ -0.01857705\},$$
$$\boldsymbol{\sigma}^{\mathrm{tr}} = \{33.60210,\ -16.80105,\ -16.80105\},$$
$$\bar{\sigma} = 50.40315,$$
$$\Delta\bar{e}^{\mathrm{p}} = 0.03370693,$$
$$\bar{e}^{\mathrm{p}} = 0.05969894,$$
$$\sigma_{\mathrm{y}} + \zeta\bar{e}^{\mathrm{p}} = 9.954841,$$
$$1 - \vartheta = 0.1975043,$$
$$\mathbf{n} = \{0.816497,\ -0.408248,\ -0.408248\},$$

$$\boldsymbol{\sigma} = \{7.946052,\ -2.008788,\ -2.008788\}\,,$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} 1.0071952 & 0 & 0 \\ 0 & 0.9960265 & 0 \\ 0 & 0 & 0.9960265 \end{bmatrix},$$

$$\Omega^t = 0.9995970\,,$$

$$\mathbf{R} = \{8.000000,\ -2.000000,\ -2.000000\}\,, \qquad \text{cf. Eq. (20)},$$

$$\mathbf{C} = \begin{bmatrix} 725.9259 & 637.0370 & 637.0370 \\ 637.0370 & 760.4832 & 602.4798 \\ 637.0370 & 602.4798 & 760.4832 \end{bmatrix}, \qquad \text{cf. Eq. (25)},$$

$$\mathbf{K} = \begin{bmatrix} 743.6009 & 638.8136 & 638.8136 \\ 638.8136 & 752.0786 & 597.4604 \\ 638.8136 & 597.4604 & 752.0786 \end{bmatrix}, \qquad \text{cf. Eq. (24)},$$

$$\varsigma = \{8.560876,\ -1.936469,\ -1.936469\}\,, \qquad \text{cf. Eq. (29)}.$$

Sensitivity analysis (h = E)

Initial values:

$$\frac{\mathrm{d}\,^t\mathbf{q}}{\mathrm{d}h} = 10^{-6} \times \{-4.159140,\ 1.708572,\ 1.708572\}\,,$$

$$\frac{\mathrm{d}\,^t\boldsymbol{\sigma}}{\mathrm{d}h} = 10^{-5} \times \{-1.429554,\ -0.227068,\ -0227068\}\,,$$

$$\frac{\mathrm{d}\,^t\bar{e}^{\mathrm{p}}}{\mathrm{d}h} = 9.282065 \times 10^{-8},$$

$$\frac{\partial K}{\partial h} = 0.666667\,, \qquad \frac{\partial(2G)}{\partial h} = 0.8\,, \qquad \frac{\partial \sigma_{\mathrm{y}}}{\partial h} + {}^t\bar{e}^{\mathrm{p}}\,\frac{\partial \zeta}{\partial h} = 0\,,$$

$$\psi_{11,1} = 0.970935\,, \qquad \psi_{22,2} = \psi_{33,3} = 1.015062\,, \qquad \text{remaining } \psi_{i\alpha,j} = 0.$$

Right-hand side vector components:

$$\frac{\partial \mathbf{Q}}{\partial h} = \{0,\ 0,\ 0\}\,,$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial h} = 10^{-3} \times \{0.9518323,\ 0.4131794,\ 0.4131794\}\,, \qquad \text{cf. Eq. (100)},$$

$$\frac{\partial \Delta\boldsymbol{\sigma}}{\partial\,^t\boldsymbol{\sigma}}\,\frac{\mathrm{d}\,^t\boldsymbol{\sigma}}{\mathrm{d}h} = 10^{-5} \times \{0.7125847,\ -0.3562924,\ -0.3562924\}\,, \qquad \text{cf. Eq. (98)},$$

$$\frac{\partial \Delta\boldsymbol{\sigma}}{\partial\,^t\bar{e}^{\mathrm{p}}}\,\frac{\mathrm{d}\,^t\bar{e}^{\mathrm{p}}}{\mathrm{d}h} = 10^{-5} \times \{0.8250725,\ -0.4125362,\ -0.4125362\}\,, \qquad \text{cf. Eq. (99)},$$

$$\frac{\mathrm{d}\overline{\mathbf{B}}}{\mathrm{d}h} = \begin{bmatrix} -\psi^2_{11,1}\,\frac{\mathrm{d}\,^tq_1}{\mathrm{d}h} & 0 & 0 \\ 0 & -\psi^2_{22,2}\,\frac{\mathrm{d}\,^tq_2}{\mathrm{d}h} & 0 \\ 0 & 0 & -\psi^2_{33,3}\,\frac{\mathrm{d}\,^tq_3}{\mathrm{d}h} \end{bmatrix}$$

$$= 10^{-6} \times \begin{bmatrix} 3.920881 & 0 & 0 \\ 0 & -1.760428 & 0 \\ 0 & 0 & -1.760428 \end{bmatrix}, \qquad \text{cf. Eq. (58)},$$

$$\frac{\mathrm{d}\overline{\overline{\mathbf{B}}}}{\mathrm{d}h}\Delta\mathbf{q} = \begin{bmatrix} -\psi^3_{11,1}\frac{\mathrm{d}\,{}^tq_1}{dh}\Delta q_1 & 0 & 0 \\ 0 & -\psi^3_{22,2}\frac{\mathrm{d}\,{}^tq_2}{dh}\Delta q_2 & 0 \\ 0 & 0 & -\psi^3_{33,3}\frac{\mathrm{d}\,{}^tq_3}{dh}\Delta q_3 \end{bmatrix}$$

$$= 10^{-6}\times\begin{bmatrix} 0.1464286 & 0 & 0 \\ 0 & 0.0330131 & 0 \\ 0 & 0 & 0.0330131 \end{bmatrix}, \quad \text{cf. Eq. (59)},$$

$$\left.\frac{\mathrm{d}\Delta\boldsymbol{\varepsilon}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = 10^{-7}\times\{1.5644416, 0.3191333, 0.3191333\}, \quad \text{cf. Eq. (61)},$$

$$\left.\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = \frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}}{\mathrm{d}h} + \frac{\partial\Delta\boldsymbol{\sigma}}{\partial h} + \frac{\partial\Delta\boldsymbol{\sigma}}{\partial\,{}^t\boldsymbol{\sigma}}\frac{\mathrm{d}\,{}^t\boldsymbol{\sigma}}{\mathrm{d}h} + \frac{\partial\Delta\boldsymbol{\sigma}}{\partial\,{}^t\bar{e}^{\mathrm{p}}}\frac{\mathrm{d}\,{}^t\bar{e}^{\mathrm{p}}}{\mathrm{d}h} + \mathbf{C}\left.\frac{\mathrm{d}\Delta\boldsymbol{\varepsilon}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)}$$

$$= 10^{-3}\times\{1.1071402, 0.5463778, 0.5463778\}, \quad \text{cf. Eqs. (96), (61)},$$

$$\frac{\mathrm{d}\tilde{\mathbf{B}}}{\mathrm{d}h} = 10^{-6}\times\begin{bmatrix} 4.213738 & 0 & 0 \\ 0 & -1.694402 & 0 \\ 0 & 0 & -1.694402 \end{bmatrix}, \quad \text{cf. Eq. (60)},$$

$$\frac{1}{\overset{*}{J}}\frac{\mathrm{d}\overset{*}{J}}{\mathrm{d}h} = -0.5696419\times10^{-6},$$

$$\left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = 10^{-3}\times\{1.1435688, 0.5485291, 0.5485291\}, \quad \text{cf. Eq. (79)}.$$

Results:

$$\frac{\mathrm{d}\Delta\mathbf{q}}{\mathrm{d}h} = \mathbf{K}^{-1}\left.\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}h}\right|_{\Delta\mathbf{q}\neq\Delta\mathbf{q}(h)} = 10^{-6}\times\{-4.496729, 1.722101, 1.722101\},$$

(incremental sensitivity solution),

$$\frac{\mathrm{d}\Delta\boldsymbol{\varepsilon}}{\mathrm{d}h} = 10^{-6}\times\{-4.372639, 1.747171, 1.747171\}, \quad \text{cf. Eq. (61)},$$

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}h} = 10^{-6}\times\{4.726735, -0.983014, -0.983014\}, \quad \text{cf. Eq. (96)},$$

$$\frac{\mathrm{d}\bar{e}^{\mathrm{p}}}{\mathrm{d}h} = 3.806500\times10^{-8}, \quad \text{cf. Eq. (97)}.$$

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}h} = 10^{-6}\times\{-4.210704, 1.780079, 1.780079\}, \quad \text{cf. Eq. (56)},$$

$$\frac{\mathrm{d}\varsigma}{\mathrm{d}h} = 10^{-5}\times\{-6.071765, -0.880797, -0.880797\}, \quad \text{cf. Eq. (81)}.$$

5. Conclusions

The algorithm for parametric sensitivity assessment for both materially and geometrically nonlinear problems of statics appears to be similar to the algorithm developed for linearized deformation case in [3]. It retains the fundamental advantages of the linear formulation of the sensitivity problem and the necessity to perform only some additional back-substitutions with the same consistent stiffness matrix previously factorized during the equilibrium iteration. The only new features that have to be addressed in the case of nonlinear geometry can be listed as follows:

- A number of additional geometric terms enter the explicit design derivative of the residual force vector R_α.

- An additional term due to the initial stress stiffness appears in the stiffness matrix.
- Similarly to stresses (and possibly other tensors needed in the equilibrium analysis), their sensitivities must be accordingly updated whenever the reference configuration is changed during the computations. The respective transformations have to be differentiated with respect to design with the design-dependence of all the geometric terms involved taken into account.

All the above features make the geometrically nonlinear sensitivity analysis more tedious and time-consuming; however, the main steps of the analysis remain the same.

The algorithm presented here retains obviously some drawbacks as compared against its geometrically linear version. We mean by it in the first place the problems of non-differentiability of the constitutive equations at the points of instantaneous stiffness change. This problem was discussed in the earlier paper [3] and is not addressed here in detail. We only recall the conclusion that such sensitivity discontinuities do not invalidate the general idea of the analysis; we only have to accept that there exist isolated time instants at which sensitivity cannot be computed uniquely.

The sensitivity formulation presented above, which includes design-dependence of the reference configurations, lends itself to relatively easy extensions towards shape sensitivity analysis. In the latter case, the initial configuration of the deformed body is assumed to explicitly depend on design. This assumption remains in close analogy to the assumption of design-dependent reference configurations in the updated Lagrangian approach and its implementation is thus straightforward.

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