Computer Assisted Mechanics and Engineering Sciences, 4: 501–519, 1997.

Trefftz-type procedure for Laplace equation on domains with circular holes, circular inclusions, corners, slits, and symmetry

Jan A. Kołodziej* and Anita Uściłowska**
* *Institute of Applied Mechanics,* ** *Institute of Mathematics*
Poznań University of Technology, Piotrowo 3, 60-965 Poznań, POLAND

(Received July 22, 1996)

The purpose of the paper is to propose of a way of constructing trial functions for the indirect Trefftz method as applied to harmonic problems possessing circular holes, circular inclusions, corners, slits, and symmetry.

In the traditional indirect formulation of the Trefftz method, the solution of the boundary-volume problem is approximated by a linear combination of the T-complete functions and some coefficients. The T-complete Trefftz functions satisfy exactly the governing equations, while the unknown coefficients are determined so as to make the boundary conditions satisfied approximately.

The trial functions, proposed and systematically constructed in this paper, fulfil exactly not only the differential equation, but also certain given boundary conditions for holes, inclusions, cracks and the conditions resulting from symmetry. A list of such trial functions, unavailable elsewhere, is presented. The efficiency is illustrated by examples in which three Trefftz-type procedures, namely the boundary collocation, least square, and Galerkin is used.

1. Introduction

The concept of the Trefftz method consists in the application of analytically derived trial functions, sometimes called T-functions, identically fulfilling certain governing differential equations inside and on the boundary of the considered area. The most popular trial functions are those known as Herrera functions or T-complete Herrera sets of functions [2-5]. For two-dimensional problems of the Laplace equation, for bounded Ω^+ and unbounded Ω^- regions, they are of the form

$$\Omega^+ - \{1, \mathrm{Re}\,(z^n)\,,\ \mathrm{Im}\,(z^n)\,;\ \ n = 1,2,3,\ldots\}\,, \tag{1}$$

$$\Omega^- - \left\{\ln\left(x^2+y^2\right), \mathrm{Re}\,(z^{-n})\,,\ \mathrm{Im}\,(z^{-n})\,;\ \ n = 1,2,3,\ldots\right\}. \tag{2}$$

The purpose of this paper is to propose a way of constructing some trial functions for application of the indirect Trefftz method to solution of harmonic problems related to domains possessing circular holes, circular inclusions, corners, slits, and symmetry. The advantage of the proposed trial functions is that they fulfil exactly not only the differential equation, but also certain boundary conditions for holes, inclusions, cracks and the conditions resulting from symmetry.

The derivation of the trial functions is based on the general solution of two-dimensional Laplace equation in polar co-ordinate system

$$\begin{aligned}\varphi(r,\theta) &= A_0 + A_1\theta + A_2\theta\ln r + \sum_{n=1}^{\infty}\left(B_n r^{\lambda_n} + C_n r^{-\lambda_n}\right)\cos(\lambda_n\theta)\\ &\quad + \sum_{n=1}^{\infty}\left(D_n r^{\lambda_n} + E_n r^{-\lambda_n}\right)\sin(\lambda_n\theta).\end{aligned} \tag{3}$$

In the classical Trefftz method, the unknown coefficients of the trial functions are calculated from a variational principle that takes into account boundary conditions on the whole boundary. In this paper three Trefftz-type procedures, namely the boundary collocation, least square, and Galerkin is used. The last one can be recognised as the traditional Trefftz method.

2. Trefftz method. Three indirect approximations

Consider the governing equation and boundary conditions in the form:

$$\nabla^2\varphi = 0 \qquad \text{in} \qquad \Omega, \tag{4}$$

$$\varphi = \overline{\varphi} \qquad \text{on} \qquad \Gamma_2, \tag{5}$$

$$\frac{\partial\varphi}{\partial n} = \overline{q} \qquad \text{on} \qquad \Gamma_2, \tag{6}$$

where ∇^2 is the two-dimensional Laplace operator, φ is the unknown function, $\frac{\partial\varphi}{\partial n}$ is the normal derivative, $\overline{\varphi}$ and $\overline{q}$ are given functions and $\Gamma = \Gamma_1 + \Gamma_2$.

The weak formulation of the boundary-value problem (4-6) can be expressed in the weighted residual form as follows

$$\int_\Omega W\nabla^2\varphi \, d\Omega + \int_{\Gamma_1} W_1(\varphi - \overline{\varphi}) \, d\Gamma + \int_{\Gamma_2} W_2\left(\frac{\partial\varphi}{\partial n} - \overline{q}\right) d\Gamma = 0, \tag{7}$$

where W, W_1 and W_2 are weighting functions.

When using the Trefftz method, the solution of the boundary value problem (4-6) is approximated by a set of complete and regular functions

$$\overline{\varphi} = \sum_{i=1}^{n} a_i N_i, \tag{8}$$

where a_i are the undetermined coefficients, and N_i are "trial" functions chosen so as to satisfy the equation

$$\nabla^2 N_i = 0. \tag{9}$$

Substituting (8) into (7), we get

$$\int_{\Gamma_1} W_1\left(\sum_{i=1}^{n} a_i N_i - \overline{\varphi}\right) d\Gamma + \int_{\Gamma_2} W_2\left(\sum_{i=1}^{n} a_i \frac{\partial N_i}{\partial n} - \overline{q}\right) d\Gamma = 0. \tag{10}$$

Depending on the selected weighting functions W_1 and W_2, one obtains various variants of the Trefftz method. In what follows, three variants that can easily be identified as the boundary collocation, least square, and Galerkin methods are discussed.

2.1. Boundary collocation method

Adopting

$$W_1 = \delta(P_j)\,, \quad P_j \in \Gamma_1\,, \quad j = 1, \ldots, m_1, \tag{11}$$

$$W_2 = \delta(Q_j)\,, \quad Q_j \in \Gamma_2\,, \quad j = 1, \ldots, m_2, \tag{12}$$

where $\delta(P_j)$ and $\delta(Q_j)$ are the Dirac delta functions, and $n = m_1 + m_2$, we get the boundary collocation method. System of linear equations for unknown coefficients a_i is in the form:

$$\sum_{i=1}^{n} a_i N_i(P_j) = \overline{\varphi}(P_j), \qquad P_j \in \Gamma_1, \tag{13}$$

$$\sum_{i=1}^{n} a_i \frac{\partial N_i(Q_j)}{\partial n} = \overline{q}(Q_j), \qquad Q_j \in \Gamma_2. \tag{14}$$

2.2. Least-square-method formulation

When W_1 and W_2 are adopted in accordance with Eq. (15),

$$W_1 = N_1 \quad \text{and} \quad W_2 = \alpha \frac{\partial N_j}{\partial n}, \quad j = 1, \ldots, n, \tag{15}$$

Eq. (10) becomes

$$\int_{\Gamma_1} N_j \left(\sum_{i=1}^{n} a_i N_i - \overline{\varphi} \right) \mathrm{d}\Gamma + \alpha \int_{\Gamma_2} \frac{\partial N_j}{\partial n} \left(\sum_{i=1}^{n} a_i \frac{\partial N_i}{\partial n} - \overline{q} \right) \mathrm{d}\Gamma = 0, \quad j = 1, \ldots, n \tag{16}$$

giving the least-square formulation of the problem.

The matrix form of Eq. (16) is

$$\mathbf{KA} = \mathbf{f}, \tag{17}$$

where

$$K_{ji} = \int_{\Gamma_1} N_j N_i \, \mathrm{d}\Gamma + \alpha \int_{\Gamma_2} \frac{\partial N_j}{\partial n} \frac{\partial N_i}{\partial n} \mathrm{d}\Gamma, \tag{18}$$

$$f_j = \int_{\Gamma_1} N_j \overline{\varphi} \, \mathrm{d}\Gamma + \alpha \int_{\Gamma_2} \frac{\partial N_j}{\partial n} \overline{q} \, \mathrm{d}\Gamma \tag{19}$$

with α being the weighting parameter that preserves the numerical equivalence between the first and the second terms of the above equation.

2.3. Galerkin formulation

Taking

$$W_1 = \frac{\partial N_j}{\partial n} \quad \text{and} \quad W_2 = -N_j, \quad j = 1, \ldots, n \tag{20}$$

and substituting them to Eq. (10) we get the Galerkin formulation:

$$\int_{\Gamma_1} \frac{\partial N_j}{\partial n} \left(\sum_{i=1}^{n} a_i N_i - \overline{\varphi} \right) \mathrm{d}\Gamma - \int_{\Gamma_2} N_j \left(\sum_{i=1}^{n} a_i \frac{\partial N_i}{\partial n} - \overline{q} \right) \mathrm{d}\Gamma = 0, \quad j = 1, \ldots, n. \tag{21}$$

Matrix form of Eq. (21) is

$$\mathbf{KA} = \mathbf{f}, \tag{22}$$

where

$$K_{ji} = \int_{\Gamma_1} \frac{\partial N_j}{\partial n} N_i \, \mathrm{d}\Gamma - \int_{\Gamma_2} N_j \frac{\partial N_i}{\partial n} \mathrm{d}\Gamma, \tag{23}$$

$$f_j = \int_{\Gamma_1} \frac{\partial N_j}{\partial n} \overline{\varphi} \, \mathrm{d}\Gamma - \int_{\Gamma_2} N_j \overline{q} \, \mathrm{d}\Gamma. \tag{24}$$

3. Construction of trial functions

3.1. Bounded region with L axes of symmetry

Consider the simply-connected region whose geometry and boundary conditions are symmetric with respect to L axes of symmetry (see Fig. 1a). We will discuss the trial functions for repeated element, bounded by two consecutive lines of symmetry, as shown in Fig. 1b.

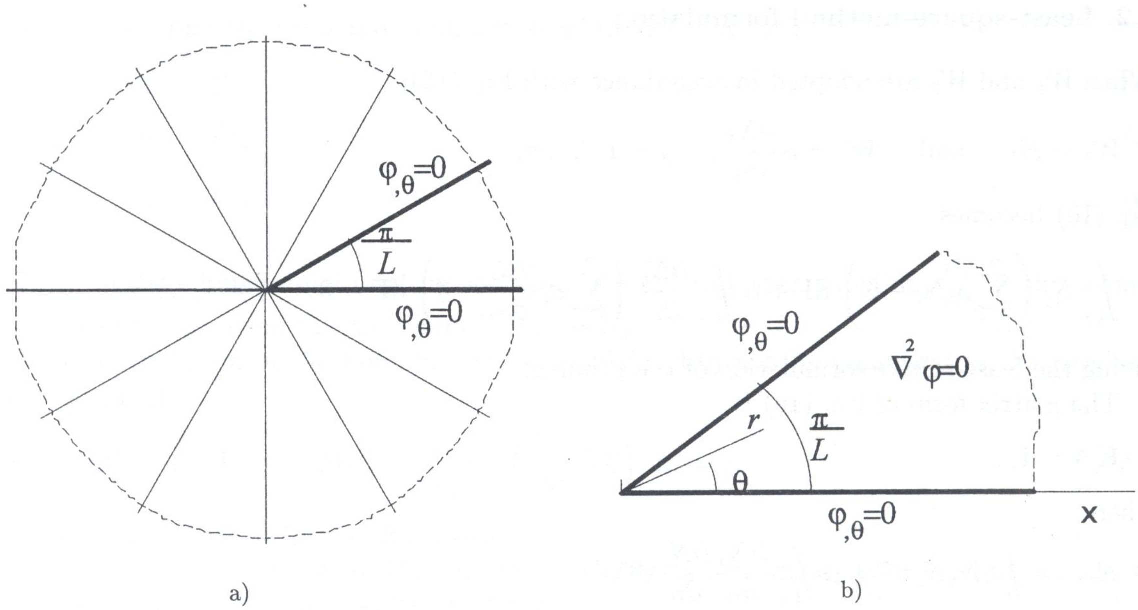

Fig. 1.

Boundary conditions resulting from symmetry have the forms

$$\frac{\partial\varphi}{\partial\theta} = 0 \qquad \text{for} \qquad \theta = 0, \tag{25}$$

$$\frac{\partial\varphi}{\partial\theta} = 0 \qquad \text{for} \qquad \theta = \frac{\pi}{L}. \tag{26}$$

Differentiating solution (3) and making use of Eq. (25), we get

$$\left.\frac{\partial\varphi}{\partial\theta}\right|_{\theta=0} = A_1 + A_2 \ln r - \sum_{n=1}^{\infty} \lambda_n \left(B_n r^{\lambda_n} + C_n r^{-\lambda_n}\right) \sin(\lambda_n 0)$$

$$+ \sum_{n=1}^{\infty} \lambda_n \left(D_n r^{\lambda_n} + E_n r^{-\lambda_n}\right) \cos(\lambda_n 0) = 0$$

from which one can deduce that

$$A_1 = D_n = E_n = 0.$$

Since the solution must be limited for $r = 0$, then

$$A_2 = C_n = 0.$$

Thus we have

$$\varphi = \sum_{n=1}^{\infty} A_n r^{\lambda_n - 1} \cos \lambda_n \theta.$$

The solution may be further specified by using (26):

$$\left.\frac{\partial\varphi}{\partial\theta}\right|_{\theta=\frac{\pi}{L}} = -\sum_{n=1}^{\infty}\lambda_n A_n r^{\lambda_n-1}\sin(\lambda_n\frac{\pi}{L}) = 0$$

from which

$$\lambda_n = Ln, \qquad \text{where} \quad n = 1,2,3,\dots$$

Hence, the final form of exact solution is

$$\varphi = \sum_{k=1}^{\infty} A_k r^{L(k-1)}\cos[L(k-1)\theta]. \tag{27}$$

After truncating the infinite series present in Eq. (27) to n first terms, we get the approximate solution

$$\tilde{\varphi} = \sum_{i=1}^{n} a_i N_i(r,\theta), \tag{28}$$

where

$$N_1 = 1,$$
$$N_i = r^{L(i-1)}\cos[L(i-1)\theta], \qquad i = 2,3,\dots$$

are the Trefftz functions for considered case.

3.2. Region with L axis of symmetry and circular central hole

Consider the two-connected region, symmetric with respect to L axes, with the circular hole as shown in Fig. 2a. Let the boundary conditions are also symmetric and, moreover, let potential on cylinder be constant. The repeated element for this region is shown in Fig. 2b.

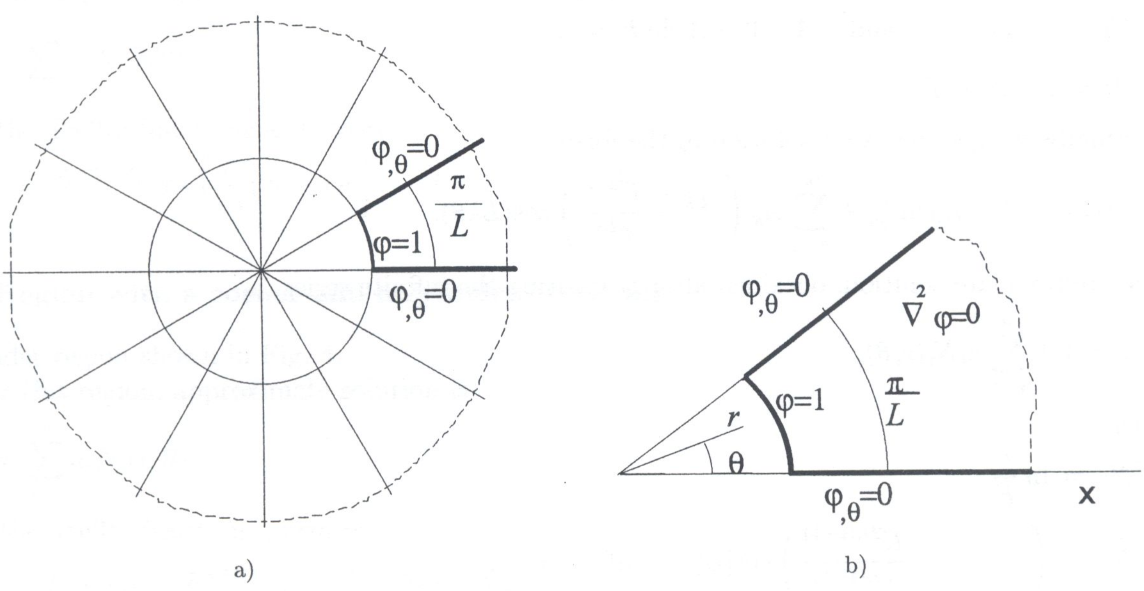

Fig. 2.

The boundary conditions resulting from the constant potential on inner cylinder, having radius E, and from the symmetry conditions are as follows

$$\varphi = 1 \qquad \text{for} \qquad r = E, \tag{29}$$

$$\frac{\partial \varphi}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \tag{30}$$

$$\frac{\partial \varphi}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = \frac{\pi}{L}. \tag{31}$$

From condition (30)

$$\left.\frac{\partial \varphi}{\partial \theta}\right|_{\theta=0} = A_1 + A_2 \ln r \sum_{n=1}^{\infty} \lambda_n \left(D_n r^{\lambda_n} + E_n r^{-\lambda_n}\right) = 0.$$

So

$$A_1 = D_n = E_n = 0.$$

From condition (31)

$$\left.\frac{\partial \varphi}{\partial \theta}\right|_{\theta=\frac{\pi}{L}} = -\sum_{n=1}^{\infty} \lambda_n \left(B_n r^{\lambda_n} + C_n r^{-\lambda_n}\right) \sin \lambda_n \frac{\pi}{L} = 0,$$

$$\lambda_n \frac{\pi}{L} = \pi n,$$

$$\lambda_n = Ln.$$

So, now

$$\varphi(r,\theta) = A_0 + A_2 \ln r + \sum_{k=1}^{\infty} \left(B_k r^{Lk} + C_k r^{-Lk}\right) \cos Lk\theta.$$

Using condition (29) we receive

$$\varphi(E,\theta) = A_0 + A_2 \ln E + \sum_{k=1}^{\infty} \left(B_k E^{Lk} + C_k E^{-Lk}\right) \cos Lk\theta = 1$$

and

$$C_k = -B_k E^{2kL} \qquad \text{and} \qquad A + 0 + A_2 \ln E = 1,$$

$$A_0 = 1 - A_2 \ln E.$$

Finally, we get the exact solution in the form

$$\varphi(r,\theta) = 1 + A_0 \ln \frac{r}{E} + \sum_{k=1}^{\infty} A_k \left(r^{Lk} - \frac{E^{2Lk}}{r^{Lk}}\right) \cos(Lk\theta). \tag{32}$$

The approximate solution, obtained after truncating the infinite series, is

$$\tilde{\varphi} = 1 + \sum_{i=1}^{n} a_i N_i(r,\theta), \tag{33}$$

where

$$N_1 = \ln \frac{r}{E},$$

$$N_i = \left(r^{L(i-1)} - \frac{E^{2L(i-1)}}{r^{L(i-1)}}\right) \cos[L(i-1)\theta], \qquad i = 2,3,4,\ldots$$

are the searched Trefftz functions for the case considered.

3.3. Region with corner

Consider the simply-connected region having the shape of an angular sector, with a vertex angle 2α, as shown in Fig. 3a. Assume that the potential on two sides of angle is equal to 0 and, that the angular sector has symmetry.

As a repeated element let us assume the wedge shown in Fig. 3b. The boundary conditions for the wedge are

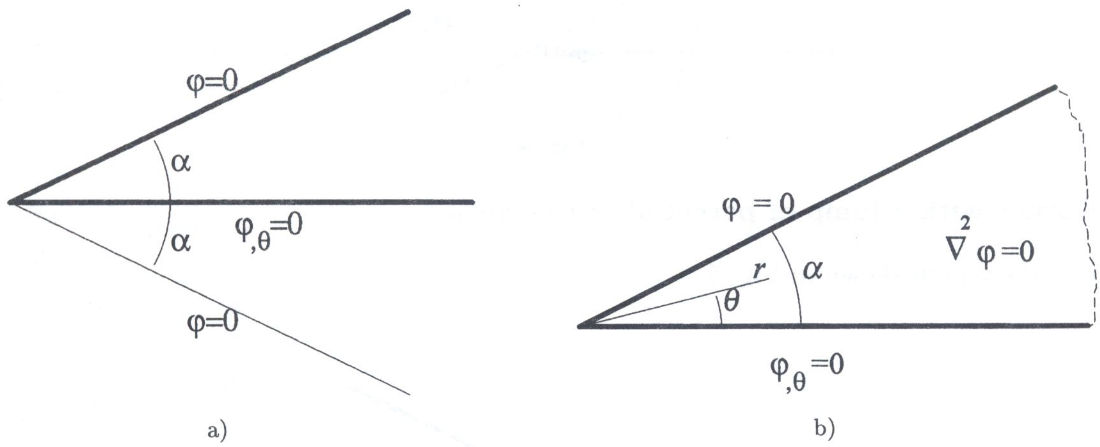

Fig. 3.

$$\frac{\partial\varphi}{\partial\theta} = 0 \qquad \text{for} \quad \theta = 0 \tag{34}$$

and

$$\varphi = 0 \qquad \text{for} \quad \theta = \alpha. \tag{35}$$

After similar considerations as in sections 3.1 and 3.2 one obtains

$$\varphi = \sum_{k=1}^{\infty} A_k r^{\frac{\pi}{2\alpha}(2k-1)} \cos\left[\frac{\pi}{2\alpha}(2k-1)\theta\right]. \tag{36}$$

Consequently, the approximate solution is of the form

$$\tilde{\varphi} = \sum_{i=1}^{n} a_i N_i(r,\theta) \tag{37}$$

with the Trefftz functions as follows

$$N_i = r^{\frac{\pi}{2\alpha}(2i-1)} \cos\left[\frac{\pi}{2\alpha}(2i-1)\theta\right], \qquad i = 1,2,3,\ldots$$

3.4. Region with a corner and a cylindrical cutting

Consider region shown in Fig. 4.

For this region, approximate solution is

$$\tilde{\varphi} = \sum_{i=1}^{n} a_i N_i(r,\theta) \tag{38}$$

with the Trefftz functions given by

$$N_i = \left(r^{\frac{\pi}{2}\frac{2i-1}{\alpha}} - \frac{E^{\pi\frac{2i-1}{\alpha}}}{r^{\frac{\pi}{2}\frac{2i-1}{\alpha}}}\right) \cos\left[\left(\frac{\pi}{2}\frac{2i-1}{\alpha}\right)\theta\right], \qquad i = 1,2,3,\ldots$$

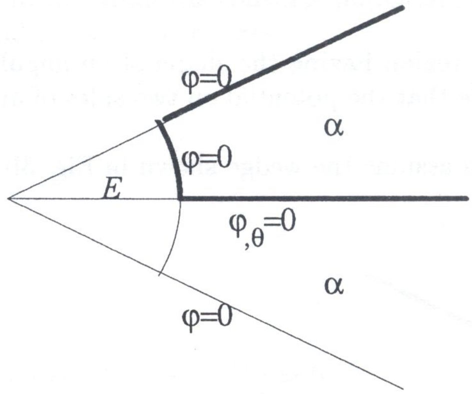

Fig. 4.

3.5. Corner with a jump of potential at the corner

Consider the region shown in Fig. 5.

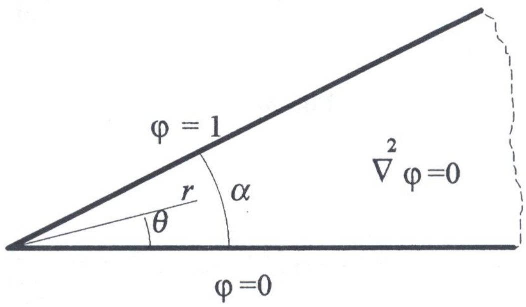

Fig. 5.

For this region, approximate solution is

$$\tilde{\varphi} = \frac{\theta}{\alpha} + \sum_{i=1}^{n} a_i N_i(r,\theta) \tag{39}$$

with the Trefftz functions $N_i(r,\theta)$ defined as follows

$$N_i = r^{\frac{2\pi}{\alpha}i} \sin\left(\frac{\pi}{\alpha} i\theta\right), \qquad i = 1,2,3,\ldots$$

3.6. Region with L axis of symmetry and a circular central hole without source inside

Consider two-connected region with L axes of symmetry. The inner boundary is circular, with radius equal to E (see Fig. 6). Assume that boundary conditions are symmetric and the potential on cylinder is constant and fulfils integral relation (43).

Boundary conditions are

$$\varphi = \varphi_0 \qquad \text{for} \qquad r = E, \tag{40}$$

$$\frac{\partial \varphi}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \tag{41}$$

$$\frac{\partial \varphi}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = \frac{\pi}{L}, \tag{42}$$

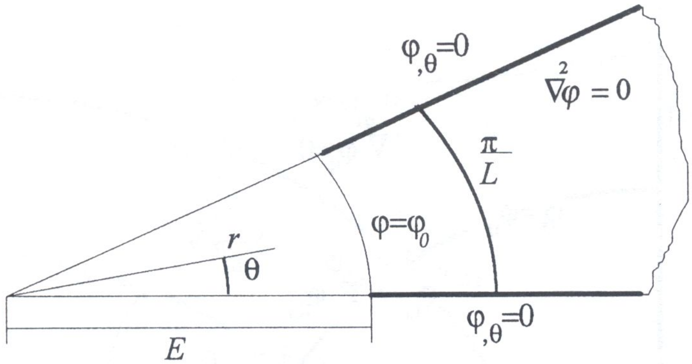

Fig. 6.

$$\int_0^{2\pi} \frac{\partial \varphi}{\partial \theta}\bigg|_{r=E} \mathrm{d}\theta = -\pi E. \tag{43}$$

The approximate solution is of the form

$$\tilde{\varphi} = \varphi_0 + \sum_{i=1}^{n} a_i N_i(r,\theta), \tag{44}$$

where the Trefftz functions are given by

$$N_i = \left(r^{Li} - \frac{E^{2Li}}{r^{Li}}\right)\cos(Li\theta), \qquad i = 1, 2, 3, \ldots$$

3.7. Region with a circular inclusion and one symmetry axis

The region of this type can be encountered when considering the thermal problem in a fibrous composite.

3.7.1. Composite with perfect contact

In the case of perfect thermal contact between a fibre and matrix, the boundary conditions are:

$$\frac{\partial \varphi_I}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \ \ 0 \le r \le E, \tag{45}$$

$$\frac{\partial \varphi_{II}}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \ \ E \le r \le 1, \tag{46}$$

$$\varphi_I = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \ \ 0 \le r \le E, \tag{47}$$

$$\varphi_{II} = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \ \ E \le r \le 1, \tag{48}$$

$$\varphi_I = \varphi_{II} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \ \ r = E, \tag{49}$$

$$\frac{\partial \varphi_I}{\partial r} = F\frac{\partial \varphi_{II}}{\partial r} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \ \ r = E. \tag{50}$$

The exact solutions which satisfy the above boundary conditions are of the form:

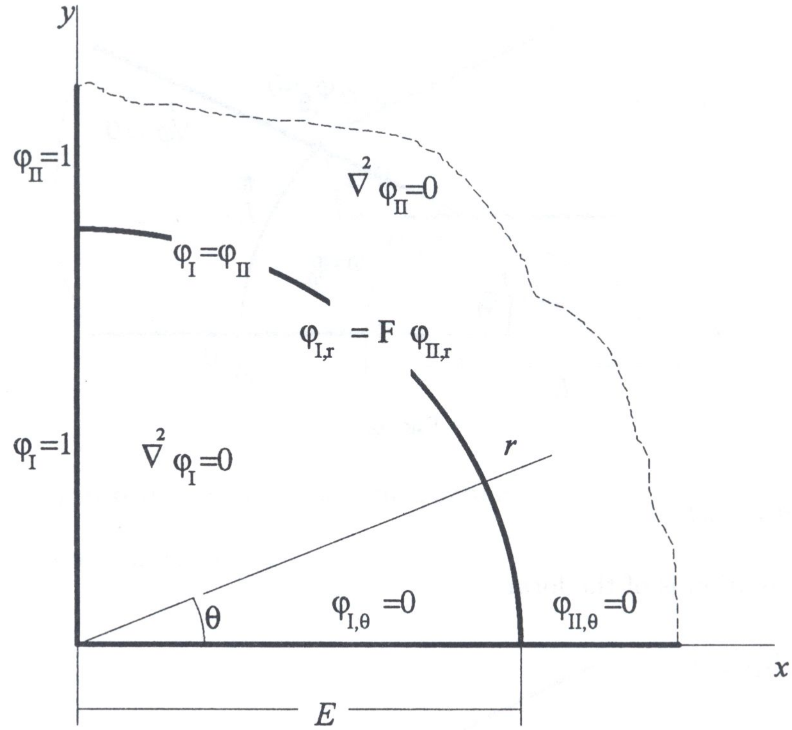

Fig. 7.

for the fibre

$$\varphi_I = 1 + \sum_{k=1}^{\infty} A_k r^{2k-1} \cos[(2k-1)\theta], \tag{51}$$

for the matrix

$$\varphi_{II} = 1 + \sum_{k=1}^{\infty} \frac{A_k}{2} \left[(1+F) r^{2k-1} + (1-F) \frac{E^{4k-2}}{r^{2k-1}} \right] \cos[(2k-1)\theta]. \tag{52}$$

The approximate solutions are

$$\tilde{\varphi}_I = 1 + \sum_{i=1}^{n} a_i N_i^I(r,\theta), \tag{53}$$

$$\tilde{\varphi}_{II} = 1 + \sum_{i=1}^{n} a_i N_i^{II}(r,\theta), \tag{54}$$

where

$$N_i^I = r^{2i-1} \cos[(2i-1)\theta], \qquad i = 1,2,3,\ldots,$$

$$N_i^{II} = \frac{1}{2} \left[(1+F) r^{2i-1} + (1-F) \frac{E^{4i-2}}{r^{2i-1}} \right] \cos[(2i-1)\theta], \qquad i = 1,2,3,\ldots$$

are the Trefftz functions for the fibre and matrix, respectively.

3.7.2. Composite of three components, all with perfect contact between constituents

The boundary conditions for this case are as follows

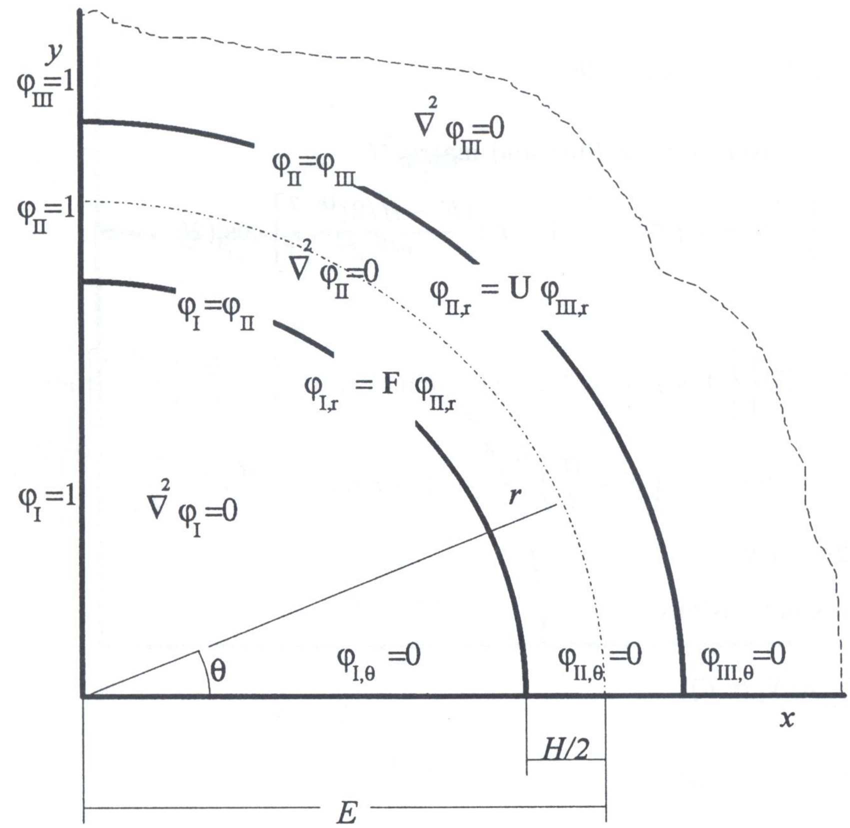

Fig. 8.

$$\frac{\partial \varphi_I}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \;\; 0 \le r \le E - H/2, \tag{55}$$

$$\frac{\partial \varphi_{II}}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \;\; E - H/2 \le r \le E + H/2, \tag{56}$$

$$\frac{\partial \varphi_{III}}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \;\; E + H/2 \le r \le 1, \tag{57}$$

$$\varphi_I = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \;\; 0 \le r \le E - H/2, \tag{58}$$

$$\varphi_{II} = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \;\; E - H/2 \le r \le E + H/2, \tag{59}$$

$$\varphi_{III} = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \;\; E + H/2 \le r \le 1, \tag{60}$$

$$\varphi_I = \varphi_{II} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \;\; r = E - H/2, \tag{61}$$

$$\varphi_{II} = \varphi_{III} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \;\; r = E + H/2, \tag{62}$$

$$\frac{\partial \varphi_I}{\partial r} = F\frac{\partial \varphi_{II}}{\partial r} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \;\; r = E - H/2, \tag{63}$$

$$\frac{\partial \varphi_{II}}{\partial r} = F\frac{\partial \varphi_{III}}{\partial r} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \;\; r = E + H/2. \tag{64}$$

The exact solutions which satisfy the above boundary conditions are of the form:

for the fibre

$$\varphi_I = 1 + \sum_{k=1}^{\infty} A_k r^{2k-1} \cos[(2k-1)\theta], \tag{65}$$

for the intermediate layer between fibre and matrix

$$\varphi_{II} = 1 + \sum_{k=1}^{\infty} \frac{A_k}{2} \left[(1+F)r^{2k-1} + (1-F)\frac{(E-H/2)^{4k-2}}{r^{2k-1}}\right] \cos[(2k-1)\theta], \tag{66}$$

for matrix

$$\begin{aligned}\varphi_{III} &= 1 + \sum_{k=1}^{\infty} \frac{A_k}{4} \left\{\left[(1+F)(1+U) + (1-F)(1-U)\left(\frac{E-H/2}{E+H/2}\right)^{4k-2}\right] r^{2k-1}\right. \\ &\quad \left. + \left[(1+F)(1-U)\left(E+\frac{H}{2}\right)^{4k-2} + (1-F)(1+U)\left(E-\frac{H}{2}\right)^{4k-2}\right] r^{-(2k-1)}\right\} \\ &\quad \cos[(2k-1)\theta]. \end{aligned} \tag{67}$$

The approximate solutions are

$$\tilde{\varphi}_I = 1 + \sum_{i=1}^{n} a_i N_i^I(r,\theta), \tag{68}$$

$$\tilde{\varphi}_{II} = 1 + \sum_{i=1}^{n} a_i N_i^{II}(r,\theta), \tag{69}$$

$$\tilde{\varphi}_{III} = 1 + \sum_{i=1}^{n} a_i N_i^{III}(r,\theta), \tag{70}$$

where

$$N_i^I = r^{2i-1}\cos[(2i-1)\theta], \qquad i = 1,2,3,\ldots,$$

$$N_i^{II} = \frac{1}{2}\left[(1+F)r^{2i-1} + (1-F)\frac{(E-H/2)^{4i-2}}{r^{2i-1}}\right]\cos[(2i-1)\theta], \qquad i = 1,2,3,\ldots,$$

$$\begin{aligned}N_i^{III} &= \frac{1}{4}\left\{\left[(1+F)(1+U) + (1-F)(1-U)\left(\frac{E-H/2}{E+H/2}\right)^{4i-2}\right] r^{2k-1}\right. \\ &\quad \left. + \left[(1+F)(1-U)\left(E+\frac{H}{2}\right)^{4i-2} + (1-F)(1+U)\left(E-\frac{H}{2}\right)^{4i-2}\right] r^{-(2i-1)}\right\} \\ &\quad \times \cos[(2i-1)\theta], \qquad i = 1,2,3,\ldots\end{aligned}$$

are the Trefftz functions for three components of the composite

3.7.3. Imperfect contact between fibre and matrix

In the case of imperfect thermal contact between a fibre and matrix, the boundary conditions are:

$$\frac{\partial \varphi_I}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \;\; 0 \le r \le E, \tag{71}$$

$$\frac{\partial \varphi_{II}}{\partial \theta} = 0 \qquad \text{for} \qquad \theta = 0, \;\; E \le r \le 1, \tag{72}$$

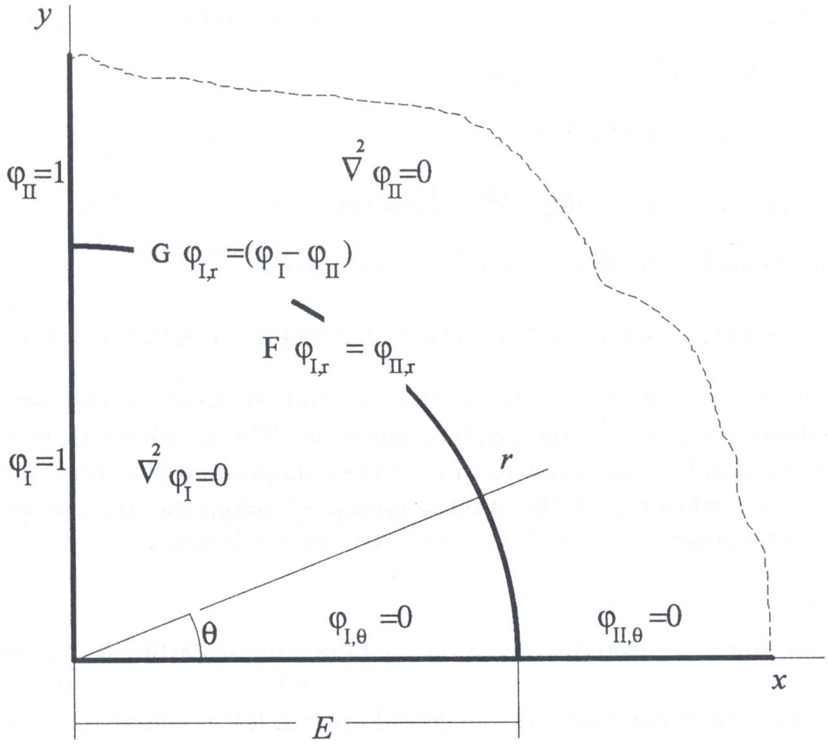

Fig. 9.

$$\varphi_I = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \quad 0 \le r \le E, \tag{73}$$

$$\varphi_{II} = 1 \qquad \text{for} \qquad \theta = \frac{\pi}{2}, \quad E \le r \le 1, \tag{74}$$

$$G\frac{\partial \varphi}{\partial r} = \varphi_{II} - \varphi_I \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \quad r = E, \tag{75}$$

$$F\frac{\partial \varphi_I}{\partial r} = \frac{\partial \varphi_{II}}{\partial r} \qquad \text{for} \qquad 0 \le \theta \le \frac{\pi}{2}, \quad r = E. \tag{76}$$

The exact solutions which satisfy the above boundary conditions are of the form:

for the fibre

$$\varphi_I = 1 + \sum_{k=1}^{\infty} A_k r^{2k-1} \cos[(2k-1)\theta], \tag{77}$$

for the matrix

$$\begin{aligned} \varphi_{II} = {} & 1 + \sum_{k=1}^{\infty} A_k \frac{1}{2} \Big\{ [1 + F - (1k-1)G]\, r^{2k-1} \\ & + E^{4k-1} [1 - F - (2k-1)G]\, r^{-(2k-1)} \Big\} \cos[(2k-1)\theta]. \end{aligned} \tag{78}$$

The approximate solutions are

$$\tilde{\varphi}_I = 1 + \sum_{i=1}^{n} a_i N_i^I(r, \theta), \tag{79}$$

$$\tilde{\varphi}_{II} = 1 + \sum_{i=1}^{n} a_i N_i^{II}(r, \theta), \tag{80}$$

where

$$N_i^I = r^{2i-1}\cos[(2i-1)\theta], \qquad i=1,2,3,\ldots,$$
$$N_i^{II} = \frac{1}{2}\Big\{[1+F-(2i-1)G]\,r^{2i-1} + E^{4i-2}\,[1-F-(2i-1)G]\,r^{-(2i-1)}\Big\}\cos[(2i-1)\theta], \qquad i=1,2,3,\ldots$$

are the Trefftz functions for the fibre and matrix, respectively.

4. Numerical examples of application of special kind of T-functions

In this section the three versions of the Trefftz method discussed above, are applied to two-dimensional problems governed by the Laplace equation. The problems include a variety of geometry ranging from simple rectangular form to curved shapes and arbitrary quadrangles.

In order to compare efficiency of the three versions, we adopt as an error index, the maximal distance between the approximate solution $\tilde{\varphi}$ and the exact solution φ:

$$ERR = \max|\tilde{\varphi}-\varphi| \tag{81}$$

The maximum error my be expected to occur on the boundary according to the maximum principle (see [1]).

For the examples considered below the maximal error ERR is found by incremental search in 200 equally distributed control points on this part of boundary where boundary condition is fulfilled approximately.

4.1. Torsion of regular polygonal bar

Consider a regular polygonal bar with L sides. A repeated element for this bar is shown in Fig. 10.

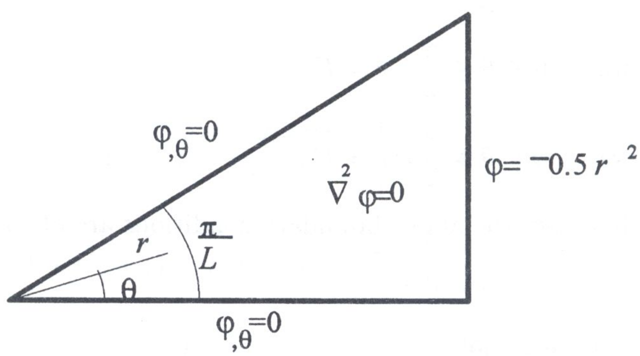

Fig. 10.

The boundary value problem for the repeated element can be formulated as follows

$$\nabla^2\varphi = 0 \qquad \text{in} \qquad 0\le x\le 1, \quad 0\le y\le x\cdot\mathrm{tg}\theta \tag{82}$$

with the boundary conditions

$$\frac{\partial\varphi}{\partial\theta} = 0 \qquad \text{for} \qquad \theta = 0, \tag{83}$$

$$\frac{\partial\varphi}{\partial\theta} = 0 \qquad \text{for} \qquad \theta = \frac{\pi}{L}, \tag{84}$$

$$\varphi = -0.5r^2 \qquad \text{for} \qquad 0\le\theta\le\frac{\pi}{L}, \quad r=\frac{1}{\cos\theta}, \tag{85}$$

where $r=(x^2+y^2)^{1/2}$, $\theta = \mathrm{arctg}(y/x)$.

In accordance with the discussion carried out in section 3.1 the approximate solution of this problem is given by

$$\tilde{\varphi} = \sum_{k=1}^{n} a_k r^{L(k-1)} \cos[L(k-1)\theta].$$

The matrices and right hand vectors for three considered methods are presented below.

A. The collocation method

$$K_{jk} = r_j^{L(k-1)} \cos[L(k-1)\theta_j], \tag{86}$$

$$f_j = -0.5 f_j^2, \tag{87}$$

where r_j, θ_j are equidistant collocation points at boundary $x = 1$.

B. The least square method

$$K_{jk} = \int_0^{\frac{\pi}{L}} \frac{\cos[L(j-1)\theta]}{(\cos\theta)^{L(j-1)}} \cdot \frac{\cos[L(k-1)\theta]}{(\cos\theta)^{L(k-1)}} \, \mathrm{d}\theta, \tag{88}$$

$$f_j = -0.5 \int_0^{\frac{\pi}{L}} \frac{\cos[L(j-1)\theta]}{(\cos\theta)^{L(j-1)+2}} \, \mathrm{d}\theta. \tag{89}$$

C. The Galerkin method

$$K_{jk} = L(j-1) \int_0^{\frac{\pi}{L}} \frac{\cos[(L(j-1)-1)\theta]}{(\cos\theta)^{L(j-1)-1}} \cdot \frac{\cos[L(k-1)\theta]}{(\cos\theta)^{L(k-1)}} \, \mathrm{d}\theta, \tag{90}$$

$$f_j = -0.5 L(j-1) \int_0^{\frac{\pi}{L}} \frac{\cos[(L(j-1)-1)\theta]}{(\cos\theta)^{L(j-1)+1}} \, \mathrm{d}\theta. \tag{91}$$

It is worth noting that the first derivative of trial function N_1 is equal to 0, which leads to singularity of the matrix $\mathbf{K}$ in formula (22). There are two ways of circumventing the difficulty caused by this fact.

The first one is to introduce a point on the boundary and find an additional equation by collocation method at this point which results in

$$K_{1k} = r_1^{L(k-1)} \cos[L(k-1)\theta_1], \tag{92}$$

where $\theta_1 = \frac{\pi}{2L}$, $r_1 = \dfrac{1}{\cos\theta_1}$.

Other elements of matrix $\mathbf{K}$ are obtained by the traditional Galerkin method.

The second way is to omit the derivative of the first trial function and to use instead the derivatives of the $2, 3, \ldots (n+1)$-th trial functions.

The results obtained by using the collocation, the least square and the Galerkin methods are presented in Tables 1 and 2.

It is clear from these Tables that the best accuracy is obtained by using least square method, while the worst accuracy corresponds to the Galerkin method with the complete Herrera T-functions.

Table 1. Maximal local error on boundary for polygon with 4 symmetry axis, n is the number of used trial functions

n	Collocation	Least square	Galerkin, complete Trefftz-functions	Galerkin, special functions	
				collocation with 1 point	$n+1$ trial functions
	ERR	ERR	ERR	ERR	ERR
3	0.3137E−02	0.5417E−02	0.2675E+00	0.3596E−02	0.7177E+00
6	0.4174E−03	0.5106E−03	0.6923E−01	0.3658E−03	0.1241E−01
9	0.1476E−03	0.1127E−03	0.5927E−01	0.7961E−04	0.1413E−02
12	0.7343E−04	0.2755E−04	0.5811E−01	0.3201E−04	0.2901E−03
15	0.4345E−04	0.1365E−03	0.5588E−01	0.1981E−04	0.2920E−04
18	0.2853E−04	0.8836E−04	0.5547E−01	0.5205E−04	0.1098E−02
21	0.2012E−04	0.2304E−04	0.5617E−01	0.3724E−04	0.1247E−03
24	0.1490E−04	0.7444E−04	0.5387E−01	0.2283E−03	0.2350E−03

Table 2. Maximal local error on boundary for polygon with 6 symmetry axis, n is the number of used trial functions

n	Collocation	Least square	Galerkin, complete Trefftz-functions	Galerkin, special functions	
				collocation with 1 point	$n+1$ trial functions
	ERR	ERR	ERR	ERR	ERR
3	0.3067E−02	0.6995E−02	0.7361E−01	0.4843E−02	0.1516E−01
6	0.9370E−03	0.1456E−02	0.6087E−01	0.1205E−02	0.6356E−01
9	0.4472E−03	0.5552E−03	0.5002E−01	0.4923E−03	0.1102E−01
12	0.2713E−03	0.2500E−03	0.4974E−01	0.2270E−03	0.3506E−02
15	0.1861E−03	0.1175E−03	0.4820E−01	0.1006E−03	0.1175E−02
18	0.1375E−03	0.9572E−04	0.4933E−01	0.8173E−04	0.5006E−03
21	0.1068E−03	0.8124E−04	0.4934E−01	0.7377E−04	0.8011E−03
24	0.8607E−04	0.8894E−04	0.4926E−01	0.8154E−03	0.9435E−03

4.2. Temperature in hollow prismatic cylinder bounded by isothermal circle and outer regular polygon

Consider a long prismatic cylinder of uniform thermal conductivity with a concentric circular hole. The adopted repeated element, as well as the formulation of its boundary value problem, are shown in Fig. 11.

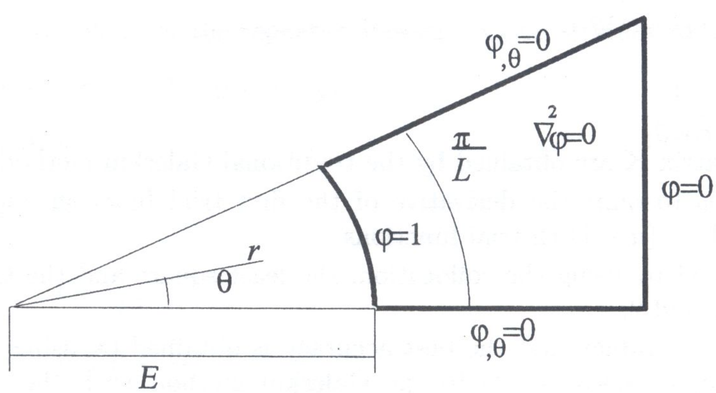

Fig. 11.

The approximate solution based on the trial functions derived in section 3.2 has the form

$$\tilde{\varphi}(r,\theta) = 1 + a_0 \ln \frac{r}{E} + \sum_{k=1}^{n} a_k \left(r^{Lk} - \frac{E^{2Lk}}{r^{Lk}} \right) \cos(Lk\theta).$$

The matrices and right-hand vectors for the three considered methods are calculated in a similar way as in the first test problem.

Results of solving this problem are shown in Tables 3 and 4.

Table 3. Maximal local error on boundary for polygon with 4 symmetry axis, and radius of inner cylinder $E = 0.8$, n is number if used trial functions

n	Collocation	Least square	Galerkin, complete Trefftz-functions	Galerkin, special functions
	ERR	ERR	ERR	ERR
3	0.1742E−01	0.1055E−01	0.44546E+00	0.9352E−02
6	0.4952E−03	0.1159E−03	0.9588E+00	0.9154E−04
9	0.2019E−04	0.2206E−05	0.1653E+00	0.1433E−05
12	0.1172E−05	0.2386E−04	0.2498E−01	0.1977E−04
15	0.8271E−07	0.6198E−03	0.1307E−01	0.1827E−03
18	0.6310E−08	0.8200E−04	0.2003E−01	0.2251E−03
21	0.5592E−09	0.3629E−04	0.3896E−01	0.2119E−03
24	0.3485E−09	0.1613E−03	0.1645E−01	0.1655E−02

Table 4. Maximal local error on boundary for polygon with 6 symmetry axis, and radius of inner cylinder $E = 0.8$, n is number if used trial functions

n	Collocation	Least square	Galerkin, complete Trefftz-functions	Galerkin, special functions
	ERR	ERR	ERR	ERR
3	0.1905E−01	0.1717E−01	0.5494E+00	0.1620E−01
6	0.4469E−02	0.3485E−02	0.2653E−00	0.3296E−02
9	0.2133E−02	0.1325E−02	0.1771E+00	0.1340E−02
12	0.1290E−02	0.5969E−03	0.1545E+00	0.6902E−03
15	0.8853E−03	0.2828E−03	0.5604E−02	0.3937E−03
18	0.6502E−03	0.2286E−03	0.7529E−02	0.2483E−03
21	0.5102E−03	0.2005E−03	0.7406E−02	0.1277E−03
24	0.4049E−03	0.2111E−03	0.7007E−02	0.1419E−03

In this case, the smallest error occurred when the boundary collocation method with equidistant collocation points was used.

The largest error was in the case of the Galerkin method with the complete T-functions, without using conditions of symmetry. The least square method in integral sense turned out to be a little better than the Galerkin method. In the last two cases special functions were used.

4.3. Motz problem

One of the most popular test problems for two-dimensional Laplace equation is the Motz problem [6].

Formulation of the relevant boundary value problem is given in Fig. 12.

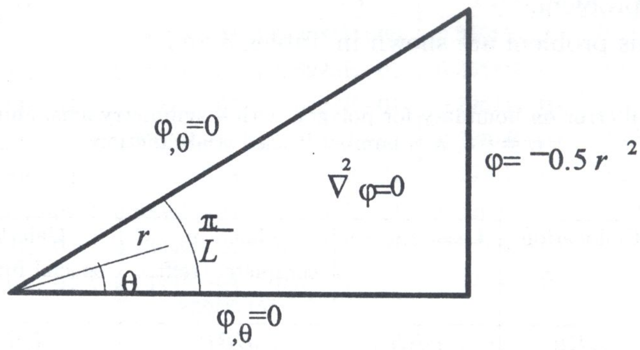

Fig. 12.

As the trial functions for the approximate solution one can be chosen those presented in section 3.3, with $\alpha = \pi$. The approximate solution is

$$\tilde{\varphi}(r,\theta) = \sum_{k=1}^{n} a_k r^{(2k-1)/2} \cos\left[\left(\frac{2k-1}{2}\right)\theta\right]. \tag{93}$$

The matrices and right-hand vectors for the three considered methods are calculated in a similar way as in the first test problem.

In this problem, because of the boundary condition we consider also the error of first derivative. This error is

$$ERRD = \max\left|\left.\frac{\partial\tilde{\varphi}}{\partial y}\right|_{y=1} - \left.\frac{\partial\varphi}{\partial y}\right|_{y=1}\right|. \tag{94}$$

Results obtained are shown in Table 5.

Table 5. Maximal local error on boundary for Motz problem, n is number if used trial functions

n	Collocation		Least square		Galerkin, – special functions	
	ERR	*ERRD*	*ERR*	*ERRD*	*ERR*	*ERRD*
3			0.5300E−01	0.1302E+00	0.2927E−01	0.1358E+00
7	0.4689E−03	0.1801E−01	0.2751E−02	0.4947E−02	0.5989E−03	0.8258E−02
11	0.1039E−04	0.1908E−03	0.3961E−03	0.3535E−03	0.4129E−04	0.1044E−02
15	0.8574E−06	0.1399E−03	0.2830E−04	0.6952E−04	0.3027E−05	0.1101E−03
19	0.9330E−07	0.3476E−04	0.1765E−05	0.8654E−05	0.2260E−06	0.1217E−04
23	0.1803E−07	0.5403E−05	0.1492E−06	0.9107E−06	0.8716E−07	0.2784E−05
27	0.3196E−08	0.8849E−06	0.2397E−06	0.2373E−06	0.2031E−06	0.6747E−05
31	0.5461E−09	0.1794E−06	0.1042E−06	0.6203E−06	0.4986E−06	0.2216E−04
35	0.9314E−10	0.2585E−07	0.4190E−05	0.3470E−05	0.1063E−04	0.4674E−04
39	0.1590E−10	0.1553E−08	0.1720E−04	0.1586E−04	0.1475E−05	0.6813E−04

5. Conclusions

A way of constructing the trial functions for the two-dimensional Laplace equation in regions possessing symmetry, holes, inclusions or angular sector has been presented. In all the considered cases the polar co-ordinate system has been used. The trial functions obtained by the proposed way fulfil exactly not only the governing equation but also some boundary conditions. These special-purpose functions can be use in hybrid Trefftz method approach, when the considered region is divided into "large elements" or in the case where one approximate solution is applied to the whole region.

The efficiency of the new special-purpose functions has been checked by solving some test problems. Solutions of the test problems have shown that the use of the special-purpose functions leads to the maximal local error of few order lesser than the error resulting from the use of the standard complete T-functions. The maximal error accompanying the calculations performed with the use of these special-purpose functions weakly depends on the version of the Trefftz method but strongly do on the number of trial functions in the approximate solution.

The test problems were solved by three versions of the Trefftz method: boundary collocation method, the Galerkin boundary method and least-square boundary method, which gave an opportunity of recognising the advantages offered by each of them. Namely, when the boundary collocation method the matrix of linear system is obtained without integration on the boundary, which essentially simplifies calculations. When the last two method are used, then the matrix of the equations for the multipliers of the trial functions is symmetric.

The research was carried out as a part of the COPERNICUS project ERB CIPA CT-940150 supported by the European Commission.

References

[1] L. Collatz. *Numerische Behandlung von Differentialgleichungen.* Springer Verlag, Berlin 1955.

[2] I. Herrera, F. Sabina. Connectivity as an alternative to boundary integral equations: Construction of bases. *Proc. Natn. Acad. Sci. USA (Appl. Math. Rhys. Sci.)*, **75**: 2059–2063, 1978.

[3] I. Herrera. *Boundary Methods: an Algebraic Theory.* Pitman Adv. Publ. Program, London 1984.

[4] H. Gurgeon, I. Herrera. *Boundary methods. C-complete systems for the biharmonic equation. Boundary Element Methods.* C.A. Brebbia ed., CML Publ., Springer, New York, 1981.

[5] I. Herrera, H. Gourgeon. Boundary methods, C-complete systems for Stokes problems. *Comp. Meth. Appl. Mech. Eng.*, **30**: 225–241, 1982.

[6] H. Motz. The treatment of singularities of partial differential equations by relaxation methods. *Quart. Appl. Math.*, **4**: 371–377, 1946.