Computer Assisted Mechanics and Engineering Sciences, 4: 369–382, 1997.


# Trefftz–Herrera Method

Ismael Herrera
*Instituto de Investigación en Matemáticas, Aplicadas y Sistemas: IIMAS*
*Universidad Nacional Autónoma de México,*
*Apartado Postal 22-582, 14000 México, D.F. México*
*e-mail: iherrera@tonatiuh.igeofcu.unam.mx*

(Received December 20, 1996)

The author's algebraic theory of boundary value problems has permitted systematizing Trefftz method and expanding its scope. The concept of TH-completeness has played a key role for such developments. This paper is devoted to revise the present state of these matters. Starting from the basic concepts of the algebraic theory, Green–Herrera formulas are presented and Localized Adjoint Method (LAM) derived. Then the classical Trefftz method is shown to be a particular case of LAM. This leads to a natural generalization of Trefftz method and a special class of domain decomposition methods: Trefftz–Herrera domain decomposition.

## 1. Introduction

By a boundary method, it is usually understood a procedure for solving partial differential equations and/or systems of such equations, in which a subregion or the entire region, is left out of the numerical treatment, by use of available analytical solutions (or, more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problems, leading to considerable economy of work and constitute a very convenient manner for treating adequately unbounded regions. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced [1, 2].

There are two main approaches to formulating boundary methods; one is based on boundary integral equations and the other one, on the use of complete systems of solutions. The author has studied extensively a version of the method based on the use of complete systems of solutions, known as Trefftz method [3–6]. Although Trefftz's original formulation was linked to a variational principle, this is not required. What is peculiar of Trefftz method, is that solutions of the homogenous differential equation — more generally, adjoint differential equation — are used as weighting functions.

The method has been used in many fields. For example, applications to Laplace's equation are given by Mikhlin [7], to the biharmonic equation by Rektorys [8] and to elasticity by Kupradze [9]. Also, many scattered contributions to the method can be found in the literature. Special mention is made here of work by Amerio, Fichera, Kupradze, Picone and Vekua [10–14]. Colton has constructed families of solutions which are complete for parabolic equations [15].

Some years ago, the author started a systematic research of Trefftz method oriented to: clarify the theoretical foundations required for using complete systems of solutions in a reliable manner, and expand the versatility of the method, making it applicable to any problem which is governed by partial differential equations and/or systems of such equations which are linear.

For symmetric systems, the results obtained were presented in several reports [5, 6, 16–24] and later integrated in book form [4]. They include: a) a criterium of completeness (introduced in [16] and called Trefftz–Herrera, or TH-completeness); b) approximating procedures and conditions for their convergence [5, 6, 18]; c) formulation of variational principles [19, 20, 24]; and d) development

of complete systems of solutions [16, 21–23]. In addition, the algebraic frame–work in which the theory has been constructed (the Algebraic Theory of Boundary Value Problems), has been used for developing biorthogonal systems of solutions [24]. Free-boundary problems, which are nonlinear even when the governing differential equations are linear, were also treated using Trefftz method [25]. Numerical procedures for fitting the boundary conditions were also discussed [26].

Function theoretic methods supply general results for developing analitically complete systems of solutions [27]. The work by Vekua in 1948 [14], by Bergman in 1961 [28] and by Gilbert in 1969 [27], and 1974 [29], on this subject was followed by many others (see, for example [30–33]). The author's algebraic theory of boundary problems permitted applying the results of function theoretic methods to specific problems; in particular the concept of TH-completeness has been quite relevant. According to Begehr and Gilbert, in their recent survey of function theoretic methods ( [34], p. 115):

> *The function theoretic approach which was pioneered by Bergman [28] and Vekua [14] and then further developed by Colton [30–32], Gilbert [27,29], Kracht–Kreyszig [33], Lanckau [35] and others, may now be effectively applied because of results of the formulation by Herrera [5] as an effective means to solving boundary value problems.*

In addition, they present many applications of the TH-completeness concept.

Many engineering applications have been made (see, for example [36–39]). A method specifically designed to deal with elastic diffraction problems was presented in [17] and later applied to answer questions of seismic engineering and seismology [40–44].

Work done after the results for symmetric operators appeared, permitted extending the algebraic theory to non-symmetric operators, leading to a generalized version of Trefftz method (Trefftz–Herrera Method [45–48]). This has been the basis of Localized Adjoint Method: LAM [49], which has been successively applied to ordinary differential equations, for which highly accurate algorithms were developed [47, 50], multidimensional steady–state problems [51], and optimal spatial methods for advection diffusion equations [52, 53].

A very successful application of LAM, to problems of transport was presented in a couple of papers [54, 55]. The resulting methodology is known as Eulerian–Lagrangian Localized Adjoint Method (ELLAM) and was developed by the ELLAM group (M.A. Celia, R.E. Ewing, T.F. Russel and the author), which was formed for this purpose. Many applications of ELLAM have been made (see, for example, [56–62], and a recent account of the subject was presented in [63].

In Herrera's generalized version of Trefftz method, the region is decomposed into many subregions, and trial and test functions which may be fully discontinuous at the internal boundaries separating such subregions, are admitted. The framework for systematic analysis supplied by the author's theory is quite useful to elucidate many questions about the performance of weighting and base functions. Indeed, since the base functions can be appropiately thought as interpolators (or extrapolators) of the actual information contained in approximate solutions, and this latter information is determined by the weighting functions that are applied, such analysis is required for choosing effective combinations of trial and test functions [64].

On the other hand, in recent years domain decomposition methods have received much attention, as a tool for solving partial differential equations. This is mainly due to the development of parallel machines, since such methods are efficient for parallelizing the numerical treatment of partial differential equations. In addition, they can be used to design adaptive algorithms which capture steep fronts that appear in many problems, such as modeling of transport. Domain decomposition methods are also used to simplify problems with complicated geometries or match regions with different physical parameters or different types of differential equations. A wealth of literature on the subjected has appeared in recent years (see for example [65–73]). Trefftz–Herrera Method in which discontinuous trial and test functions are admitted, leads in a direct manner to domain decomposition procedures. For example, the procedure presented in [51] is essentially a domain decomposition method. However, it was only recently that research on Trefftz–Herrera Method as a route to domain decomposition procedures, was initiated [74].

Here, an overview of Trefftz–Herrera Method is presented. Sections 2 and 3, are devoted to the abstract formulation of domain decomposition, which supplies the framework for this very general version of TH–Method, applicable to any linear partial differential equation, or system of such equations — see [54].

In Sections 4 to 7, applications to general differential equations of second order and elliptic type, are developed in greater detail. For the symmetric and positive definite case, the matrices of the resulting domain decomposition algorithms are also symmetric and positive definite. This property allows a direct application of efficient iterations schemes, such as the conjugate gradient method. Also, an alternative formulation of collocation procedures, "TH–collocation", which enjoys more relaxed continuity conditions than "standard collocation [75].

## 2. Abstract formulation of domain decomposition

In this section some of the most basic concepts and results of the author's algebraic theory of boundary value problems, are presented. Generally, proofs are not included. Some of these concepts and results were introduced in [45–48]. They also imply a kind of operator extensions whose connection with the theory of distributions was discussed in [76].

The discussions refer to linear operators of the type $P : D_1 \longrightarrow D_2^*$, whose domain is a the linear space $D_1$, and whose values are linear functionals on $D_2$ (i.e.; elements of $D_2^*$: the algebraic dual of the linear space $D_2$), as well as to operators whose domain is $D_2$ and with values in $D_1^*$. In particular, the transpose of $P : D_1 \longrightarrow D_2^*$ is $P^* : D_2 \longrightarrow D_1^*$.

The notation $< Pu, v >$ is used to denote the value of the functional $Pu$ at $v \in D_2$. Clearly, $< Pu, v >$ is bilinear and this defines a one-to-one correspondence between operators $P : D_1 \longrightarrow D_2^*$ and bilinear forms $< Pu, v >$ on $D_1 \oplus D_2$.

**Definition 1** *Boundary Operators.*

$B$ is a boundary operator for $P$, iff

$$< Pu, v >= 0\,, \quad \forall v \in N_{B^*} \Rightarrow Pu = 0\,. \tag{1}$$

**Definition 2** *Formal Adjoints.*

Two operators $P : D_1 \longrightarrow D_2^*$ and $Q : D_2 \longrightarrow D_1^*$ are formal adjoints when $S \equiv P - Q^*$ is a boundary operator for $P$, while $S^* \equiv P^* - Q$ is a bounadry operator for $Q$.

**Definition 3** *The subspaces $I_P$ and $I_Q$.*

Let $P : D_1 \longrightarrow D_2^*$ and $Q : D_2 \longrightarrow D_1^*$ be formal adjoints, then the subspace $I_P \subset D_1$ is defined by

$$I_P = N_P + N_S \tag{2a}$$

and the subspace $I_Q \subset D_2$ by

$$I_Q = N_Q + N_{S^*} \tag{2b}$$

Here, as before, $S \equiv P - Q^*$.

**Definition 4** *TH-completeness.*

A subset $\mathcal{W} \subset I_Q \subset D_2$, is said to be TH-complete when for any $V \in D_1$ one has

$$< SV, w >= 0\,, \quad \forall w \in \mathcal{W} \Rightarrow V \in I_P\,. \tag{3a}$$

Similarly, a subset $\mathcal{W} \subset I_P \subset D_1$, is said to be TH-complete when for any $W \in D_2$ one has

$$< S^*W, v >= 0\,, \quad \forall v \in \mathcal{W} \Rightarrow W \in I_Q\,. \tag{3b}$$

**Definition 5** *Disjoint Operators.*

A pair of operators $\{R_1, R_2\}$ of the same kind is said to be disjoint when $R_1$ is a boundary operator for $R_2$, while $R_2$ is a boundary operator for $R_1$.

A system of operators $\{R_1, R_2, \ldots, R_n\}$ of the same kind is said to be disjoint when each pair $\{R_i, R_j\}$, with $i \neq j$, is disjoint.

**Definition 6** *Completely disjoint.*

A pair of operators $\{R_1, R_2\}$ of the same kind are said to be completely (or fully) disjoint when, in addition to being disjoint, the pair $\{R_1^*, R_2^*\}$ is also disjoint.

A system of operators $\{R_1, R_2, \ldots, R_n\}$ of the same kind is said to be fully disjoint when each pair $\{R_i, R_j\}$, with $i \neq j$, is fully disjoint.

**Definition 7** *Decomposition* $\{R_1, R_2\}$ *of* $R$.

A pair of operators $\{R_1, R_2\}$ is said to be a decomposition of $R$, when they are completely disjoint and

$$R = R_1 + R_2 \,. \tag{4}$$

A system of operators $\{R_1, R_2, \ldots, R_n\}$ is said to be a decomposition of $R$, when they are fully disjoint and

$$R = R_1 + \ldots + R_N \,. \tag{5}$$

**Proposition 1** *Assume the pair* $\{R_1, R_2\}$ *decomposes* $R$, *then*

$$N_R = N_{R_1} \cap N_{R_2} \,. \tag{6}$$

**Remark 1** *If* $\{R_1, R_2\}$ *decomposes* $R$, *then* $\{R_1^*, R_2^*\}$ *decomposes* $R^*$.

**Definition 8** *Green–Herrera formula.*

The equation

$$P - B = Q^* - C^* \tag{7}$$

is said to be a Green–Herrera formula for the pair $\{P, Q\}$, when $P$ and $Q$ are formal adjoints and the pair $\{B, -C^*\}$ decomposes $S = P - Q^*$, while $\{B^*, -C\}$ decomposes $S^* = P^* - Q$.

**Theorem 1** *Assume Eq. 5 is satisfied and it is a Green–Herrera formula for the pair* $\{P, Q\}$. *Let* $\{B_1, B_2\}$ *and* $\{C_1, C_2\}$, *be decompositions of* $B$ *and* $C$, *respectively. Then, the equation*

$$(P - B_1) - B_2 = (Q - C_1)^* - C_2^* \tag{8}$$

*is a Green–Herrera formula for the pair* $\{(P - B_1), (Q - C_1)\}$.

**Definition 9** *The (abstract) boundary value problems.*

Let $B$ be a boundary operator for $P$. Given $U \in D_1$ and $V \in D_1$, the abstract boundary problem consists in finding $u \in D_1$ such that

$$Pu = f \quad \text{and} \quad Bu = g \,, \tag{9a}$$

where $f = PU \in D_2^*$ and $g = BV \in D_2^*$.

Similarly, given $W \in D_2$ and $Y \in D_2$, the adjoint boundary value problem consists in finding $w \in D_2$, such

$$Qw = QW \quad \text{and} \quad Cw = CY \,. \tag{9b}$$

**Theorem 2** *Variational formulation in terms of the data.*

*$u \in D_1$, is solution of the boundary problem, iff*

$$(P - B)\, u = f - g\,. \tag{10}$$

**Theorem 3** *Variational formulation in terms of the sought information.*

*When $P - B = Q^* - C^*$ is a Green's formula, $u \in D_1$, is solution of the boundary problem, iff*

$$(Q^* - C^*)\, u = f - g\,. \tag{11}$$

**Definition 10** *The kernel of the B.V.P., and uniqueness.*

The set $N \subset D_1$, defined by

$$N = N_P \cap N_B \tag{12}$$

is the kernel of the B.V.P. When $N = \{0\}$, the B.V.P. is said to satisfy uniqueness.

**Definition 11**

Given $C : D_2 \longrightarrow D_1^*$ and a set $\mathcal{W} \subset D_2$, the set $C\mathcal{W} \subset D_1^*$ is said to span the range of $C$, when for any $v \in D_1$, one has

$$< C^* v, w >= 0, \quad \forall w \in \mathcal{W} \Rightarrow C^* v = 0\,. \tag{13}$$

**Theorem 4** *Let $P - B = Q^* - C^*$ be a Green's formula and $\mathcal{W} \subset N_Q \subset D_2$ be a set spanning $C$. Assume a solution $u \in D_1$ of the b.v.p. exists, and let $\widehat{u} \in D_1$ be any element of $D_1$. Then, the following assertions are equivalent:*

i. $$< C^* \widehat{u}, w >= - < f - g, w >\,, \quad \forall w \in \mathcal{W}\,, \tag{14}$$

ii. $$< C^* \widehat{u}, w >=< C^* u, w >, \quad \forall w \in \mathcal{W}\,, \tag{15}$$

iii. $$C^* \widehat{u} = C^* u\,, \tag{16}$$

iv. *If $\widehat{U} \in D_1$ satifies $P\widehat{U} = f$, then*

$$< C^* \widehat{u}, w >=< (C^* \widehat{U} - B\widehat{U} + g), w >\,, \quad \forall w \in \mathcal{W}\,. \tag{17}$$

**Remark 2** *In particular, when $C^* \widehat{U} = 0$, then Eq. (16) becomes*

$$< C^* \widehat{u}, w >= - < (B\widehat{U} - g), w >\,, \quad \forall w \in \mathcal{W}\,. \tag{18}$$

*Proof:* Observe that when $w \in \mathcal{W} \subset N_Q$, one has

$$- < C^* \widehat{u}, w >=< f - g, w >=< (P - B)u, w >=< (Q^* - C^*)u, w >= - < C^* u, w >\,. \tag{19}$$

Hence (14) implies (15), which in turn implies (16), by virtue of Definition 11. From Eq. (19), it follows that when $P\widehat{U} = f$ and Eq. (16) holds, one has

$$\begin{aligned} - < C^* \widehat{u}, w > &= < P\widehat{U} - g, w >=< (Q^* - C^* + B)\widehat{U}, w > - < g, w > \\ &= < (-C^* + B)\, \widehat{U}, w > - < g, w > \end{aligned} \tag{20}$$

and Eq. (17) is clear. Using Eq. (20), one can derive (14) from (18).

## 3. INITIAL-BOUNDARY VALUE PROBLEMS WITH PRESCRIBED JUMPS

Consider a region $\Omega$ and a partition $\{\Omega_1, \ldots, \Omega_N\}$ of $\Omega$ into subregions $\Omega_1, \ldots, \Omega_N$. The linear spaces $D_1$ and $D_2$ of trial and test functions respectively, defined in $\Omega$, whose elements may have jump discontinuities across some internal boundaries whose union will be denoted by $\Sigma$ (Fig. 1). For example, in applications of the theory to finite element methods, the set $\Sigma$ could be the union of all the interelement boundaries.

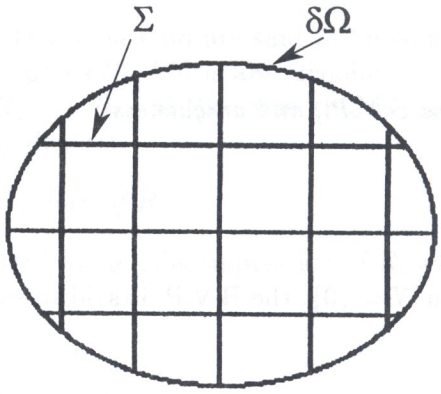


Fig. 1 The region $\Omega$, its boundary $\partial\Omega$ and internal boundaries $\Sigma$

In this setting the general boundary value problem to be considered is one with prescribed jumps, across $\Sigma$. The differential equation is

$$\mathcal{L}u = f_\Omega \quad \text{in}\,\Omega\,, \tag{21}$$

where $\Omega$ may be a purely spatial region or, more generally, a space–time region. Certain boundary and jump conditions are specified on the boundary $\partial\Omega$ and on the internal boundaries $\Sigma$, respectively. When $\Omega$ is a space–time region, such conditions generally include initial conditions. In the literature on mathematical modeling of macroscopic physical systems, there is a large class of problems that can be formulated as initial-boundary value problems with prescribed jumps or boundary value problems, when steady states are considered. For example, problems of elastic wave diffraction can be formulated as such, [17]. The jump conditions to be satisfied across $\Sigma$ by the sought solution, depend on the specific application and on the differential operator considered. For elliptic problems of second order, the jump of the function and its normal derivative are usually prescribed. Frequently, continuity of the sought solution and its normal derivative, is required — this corresponds to prescribing zero jumps for the solution and its first derivatives. When the partial differential equations mimic continuous systems, the jump conditions can be derived systematically, from the balance equations of continuum mechanics [77].

The definition of formal adjoint requires that a differential operator $\mathcal{L}$ and its formal adjoint $\mathcal{L}^*$, satisfy the condition that $w\mathcal{L}u - u\mathcal{L}^*w$ be a divergence; i.e.:

$$w\mathcal{L}u - u\mathcal{L}^*w = \nabla \cdot \{\underline{\mathcal{D}}\,(u, w)\} \tag{22}$$

for a suitable vector-valued bilinear function $\underline{\mathcal{D}}\,(u, w)$, and a general Green–Herrera formula is

$$\begin{aligned} &\int_\Omega w\mathcal{L}u\,\mathrm{d}x - \int_{\partial\Omega} \mathcal{B}\,(u, w)\,\mathrm{d}x - \int_\Sigma \mathcal{J}\,(u, w)\,\mathrm{d}x \\ &\qquad = \int_\Omega u\mathcal{L}^*w\,\mathrm{d}x - \int_{\partial\Omega} \mathcal{C}\,(w, u)\,\mathrm{d}x - \int_\Sigma \mathcal{K}\,(w, u)\,\mathrm{d}x\,. \end{aligned} \tag{23}$$

Here, integrals over the region $\Omega$, are understood as sums of integrals over the individual regions $\Omega_i$. Thus, by definition:

$$\int_\Omega w\mathcal{L}u\,\mathrm{d}x = \sum_i^N \int_{\Omega_i} w\mathcal{L}u\,\mathrm{d}x \tag{24}$$

and such integrals are well defined in spite of the fact that the differential operator is not defined on $\Sigma$. In addition,

$$\mathcal{J}(u,w) \equiv -\underline{\mathcal{D}}([u],\dot{w})\cdot\underline{n}\,, \tag{25a}$$

$$\mathcal{K}^*(u,w) \equiv \mathcal{K}(w,u) = \underline{\mathcal{D}}(\dot{u},[w])\cdot\underline{n}\,, \tag{25b}$$

where

$$[u] = u_+ - u_-, \quad \dot{u} = (u_+ + u_-)/2 \tag{26}$$

and $\mathcal{B}(u,w)$ and $\mathcal{C}(w,u) = \mathcal{C}^*(u,w)$ are two bilinear functions with the property that

$$\underline{\mathcal{D}}(u,w)\cdot\underline{n} = \mathcal{B}(u,w) - \mathcal{C}^*(u,w)\,, \tag{27}$$

the definition of $\mathcal{B}(u,w)$ depend on the type of boundary and initial conditions to be prescribed. A basic property required of $\mathcal{B}(u,w)$ is that for any $u$ that satisfies the prescribed boundary and initial conditions, $\mathcal{B}(u,w)$ is a well defined linear function of $w$, independent of the particular choice of $u$.

In order to relate these developments to the general framework of Section 2, define

$$< Pu,w > = \int_\Omega w\mathcal{L}u\,\mathrm{d}x - \int_{\partial\Omega}\mathcal{B}(u,w)\,\mathrm{d}x; \qquad < Bu,w > = \int_\Sigma \mathcal{J}(u,w)\,\mathrm{d}x\,, \tag{28}$$

$$< Qw,u > = \int_\Omega u\mathcal{L}^*w\,\mathrm{d}x - \int_{\partial\Omega}\mathcal{C}(w,u)\,\mathrm{d}x; \qquad < Cw,u > = \int_\Sigma \mathcal{K}(w,u)\,\mathrm{d}x \tag{29}$$

and the linear functionals $f$, $g\varepsilon D_2^*$ by means of:

$$< f,w > = \int_\Omega w f_\Omega\,\mathrm{d}x - \int_{\partial\Omega}\mathcal{B}(u',w)\,\mathrm{d}x; \quad \forall w\in D_2\,, \tag{30a}$$

$$< g,w > = \int_\Sigma \mathcal{J}(u''w)\,\mathrm{d}x; \quad w\in D_2\,, \tag{30b}$$

where $u', u''\in D_1$, are two auxiliary functions used to prescribe the boundary and jump conditions, respectively. Then, a formulation suitable for initial-boundary value problems with prescribed jumps, is given by Definition 9; i.e.:

$$Pu = f; \quad Bu = g\,. \tag{31}$$

## 4. Elliptic Equations

The developments thus far presented, are applicable not only to differential equations, but also to systems of such equations [54]. However, in what follows attention will be restricted to the differential equation associated with the most general elliptic operator of second order which will be written as:

$$\mathcal{L}u \equiv -\nabla\cdot\left(\underline{\underline{\mathbf{a}}}\cdot\nabla u\right) + \nabla\cdot(\underline{\mathbf{b}}u) + \mathbf{c}u = f_\Omega\,, \tag{32a}$$

for which

$$\mathcal{L}^*w \equiv -\nabla\cdot\left(\underline{\underline{\mathbf{a}}}\cdot\nabla w\right) - \underline{\mathbf{b}}\cdot\nabla w + \mathbf{c}w \tag{32b}$$

and

$$\underline{\mathcal{D}}(u,w) = \underline{\underline{\mathbf{a}}}\cdot(u\nabla w - w\nabla u) + \underline{\mathbf{b}}uw\,. \tag{33}$$

To be specific, only Dirichlet type boundary conditions will be considered, and then one can define:

$$\mathcal{B}(u,w) = \left(\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w + \mathbf{b}_n w\right) u \quad \text{and} \quad \mathcal{C}^*(u,w) = w\left(\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u\right), \tag{34}$$

where $\mathbf{b}_n = \underline{\mathbf{b}}\cdot\underline{n}$. In some applications, the complementary boundary values $\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u$, can be interpreted as diffussive flux, but other interpretations are feasible. In previous articles, fully discontinuous trial and test functions have been considered, but for simplicity, here only continuous trial and test functions — with possibly discontinuous first derivatives — will be treated. Then:

$$\mathcal{J}(u,w) = w\left[\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u\right] \quad \text{and} \quad \mathcal{K}^*(u,w) = \left[\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w\right] u\,. \tag{35}$$

When $\mathcal{L}$ is the elliptic operator of Eqs. (30), and the boundary and jump conditions are

$$u = u_\partial, \text{ on } \partial\Omega;\ \left[\underline{\underline{\mathbf{a}}}\cdot\nabla u\right]\cdot\underline{n} = j \quad \text{on } \Sigma\,, \tag{36}$$

where $u_\partial$ and $j$ are prescribed functions defined on $\partial\Omega$ and on $\Sigma$, respectively. As mentioned previously, a physical interpretation of $j$ is the jump of diffusive flux.

The boundary value problem with prescribed jumps can be formulated variationally, point-wise on $\partial\Omega$ and on $\Sigma$, by

$$\mathcal{B}(u,w) = \mathcal{B}(u_\partial, w) \quad \text{and} \quad \mathcal{J}(u,w) = \mathcal{J}(j,w)\,; \quad \forall w \in D_2\,, \tag{37}$$

respectively; or, more explicitly:

$$\left(\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w\right) u = \left(\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w\right) u_\partial \quad \text{and} \quad w\left[\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u\right] = wj\,. \tag{38}$$

The variational formulations of previous sections can be applied to this problem, introducing the linear functionals $f$ and $g\epsilon D_2^*$, defined by means of:

$$< f, w > = \int_\Omega w f_\Omega \,\mathrm{d}x - \int_{\partial\Omega} u_\partial \left(\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w\right) \mathrm{d}x; \quad < j, w > = \int_\Sigma jw \,\mathrm{d}x\,. \tag{39}$$

However, to simplify the presentation, in what follows $j \equiv 0$ will be assumed; i.e., for the sought solution, the diffusive flux will be required to be continuous across $\Sigma$.

## 5. Variational formulation for elliptic equations

The approach to domain decomposition methods based on the use of specialized weighting functions which concentrate the information, contained in approximate solutions, on the internal boundaries of the subdomains in which the original domain has been decomposed — functions possessing this property are those belonging to the set $N_Q$, considered in Section 2 — can be formulated applying the variational principles of Theorem 4 (Section 2). In this section, TH-Domain Decomposition Methods for the elliptic differential equation of Section 4, are discussed.

When $Q : D_1 \longrightarrow D_2^*$, is defined by Eq. (29), while $\mathcal{L}^*$ and $\mathcal{C}$ are those of Section 4 — Eqs. (32b) and (34), respectively — functions $w \in N_Q$, are characterized by:

$$\mathcal{L}^* w \equiv -\nabla\cdot\left(\underline{\underline{\mathbf{a}}}\cdot\nabla w\right) - \underline{\mathbf{b}}\cdot\nabla w + \mathbf{c}w = 0 \quad \text{in } \Omega \tag{40}$$

together with

$$w = 0 \quad \text{on } \partial\Omega\,. \tag{41}$$

When use is made of Eqs. (34) and (35), the variational Eq. (14) of Section 2, can be written as:

$$-\int_\Sigma \left[\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w\right] u \,\mathrm{d}x = \int_\Omega w f_\Omega \,\mathrm{d}x - \int_{\partial\Omega} u_\partial \left(\underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w\right) \mathrm{d}x, \quad \forall w \in N_Q\,. \tag{42}$$

Introducing the notation $\mathbf{a}_n \equiv \underline{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\underline{n}$ and $\partial w/\partial n$ for the normal derivative, this equation reduces to:

$$-\int_\Sigma \mathbf{a}_n\left[\partial w/\partial n\right] u\,\mathrm{d}x = \int_\Omega w f_\Omega\,\mathrm{d}x - \int_{\partial\Omega} u_\partial \mathbf{a}_n \partial w/\partial n\,\mathrm{d}x, \quad \forall w \in N_Q \tag{43}$$

when Eq. (41) and the continuity of test functions are taken into account.

The bilinear functional of the left-hand side in Eq. (43), has special properties when the differential operator $\mathcal{L}$ of Eq. (32a) is symmetric (i.e.,$\mathcal{L} \equiv \mathcal{L}^*$) and positive definite. This corresponds to taking $D_1 = D_2 = D$, $\underline{\underline{\mathbf{a}}}$ positive definite, $\underline{\mathbf{b}} \equiv 0$, $\mathbf{c} \geq 0$. In this case, given any two functions $\widehat{u}_H$ and $w \in N_Q = N_P$ (i.e., which satisfy Eqs. (40) and (41)), we have

$$-\int_\Sigma \mathbf{a}_n\left[\partial w/\partial n\right]\widehat{u}_H\,\mathrm{d}x \equiv \int_\Omega \left\{\left(\underline{\underline{\mathbf{a}}}\cdot\nabla w\right)\cdot\nabla\widehat{u}_H + \mathbf{c}w\widehat{u}_H\right\}\,\mathrm{d}x \tag{44}$$

which exhibits the bilinear functional of the left-hand side in Eq. (43), as symmetric and positive definite when restricted to functions belonging to $N_Q = N_P$. Eq. (44), supplies two expressions for this functional and its quadratic form.

Since the quadratic form is positive definite, it is possible to associate a minimum principle for the boundary solutions of the problem we have been considering. Because of the identity (44), it has two alternative formulations; the corresponding functionals are

$$\begin{aligned}\mathfrak{F}_1\left(\widehat{u}_H\right) \equiv & -\int_\Sigma \widehat{u}_H \mathbf{a}_n\left[\partial\widehat{u}_H/\partial n\right]\cdot\underline{n}\,\mathrm{d}\underline{x} \\ & -2\left\{\int_\Omega \widehat{u}_H f_\Omega d\underline{x} - \int_{\partial\Omega} u_\partial\left(\mathbf{a}_n\partial\widehat{u}_H/\partial n\right)\,\mathrm{d}\underline{x}\right\}\end{aligned} \tag{45a}$$

and

$$\begin{aligned}\mathfrak{F}_2\left(\widehat{u}_H\right) \equiv & \int_\Omega \left\{\nabla\widehat{u}_H\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla\widehat{u}_H + \mathbf{c}\widehat{u}_H^2\right\}\,\mathrm{d}\underline{x} \\ & -2\left\{\int_\Omega \widehat{u}_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial\left(\mathbf{a}_n\partial\widehat{u}_H/\partial n\right)\cdot\underline{n}\,\mathrm{d}\underline{x}\right\}.\end{aligned} \tag{45b}$$

**Under the assumption that:**

*Given any element $\widehat{u} \in D$, there is an element $\widehat{u}_H \in N_Q = N_P$, such that*

$$\widehat{u}_H = \widehat{u} \quad \text{on } \Sigma. \tag{46}$$

**The minimum principle states that:**

*Let $u \in D$, be the solution of the boundary value problem. Then either one of these functionals attains its minimum on $N_Q$, if and only if, $\widehat{u}_H = u$ on $\Sigma$.*

*Proof:* It follows from the fact that

$$\mathfrak{F}_1\left(\widehat{u}_H\right) \equiv -\int_\Sigma \widehat{u}_H \mathbf{a}_n\left[\partial\widehat{u}_H/\partial n\right]\cdot\underline{n}\,\mathrm{d}\underline{x} + 2\int_\Sigma u\mathbf{a}_n\left[\partial\widehat{u}_H/\partial n\right]\cdot\underline{n}\,\mathrm{d}\underline{\mathrm{x}}$$

by virtue of the variational equation (43). Take $u_H \in N_Q$ such that $u_H = u$ on $\Sigma$. Then

$$\mathfrak{F}_1\left(\widehat{u}_H\right) \equiv -\int_\Sigma \left(\widehat{u}_H - u_H\right)\mathbf{a}_n\left[\partial\left(\widehat{u}_H - u_H\right)/\partial n\right]\cdot\underline{n}\,\mathrm{d}\underline{x} - \int_\Sigma u_H\mathbf{a}_n\left[\partial u_H/\partial n\right]\cdot\underline{n}\,\mathrm{d}\underline{x}. \tag{47}$$

Clearly

$$\mathfrak{F}_1\left(\widehat{u}_H\right) \geq -\int_\Sigma u_H\mathbf{a}_n\left[\partial u_H/\partial n\right]\cdot\underline{n}\,\mathrm{d}\underline{x} \geq 0$$

and the equality sign holds, if and only if $\widehat{u}_H \equiv u_H$, on $\Sigma$.

## 6. THE TEST AND TRIAL FUNCTIONS

For simplicity, consider a rectangular region decomposed into rectangles (Fig. 2a). With each node $(x_i, y_j)$ — $i = 0, 1, \ldots, E_x$, and $j = 0, 1, \ldots, E_y$ — a subregion $(\Omega_{ij})$ is associated. When the node is internal, this is the union of the four rectangles that surround the node. When the node lies on the boundary $\partial\Omega$, it is the union of those rectangles inside the region, only. Observe that the total number of subregions is $E = (E_x + 1) \times (E_y + 1)$ and that the system of subregions $\{\Omega_{ij}\}$ is overlapping.

The boundary of $\Omega_{ij}$ is $\partial\Omega_{ij}$, and that part of $\Sigma$ laying in the interior of $\Omega_{ij}$ will be denoted by $\Sigma_{ij}$ (Fig. 2b). In the case of interior nodes, $\Sigma_{ij}$ is constituted by four segments and it is cross–shaped. Those four segments will be numbered as indicated in Fig. 2b.

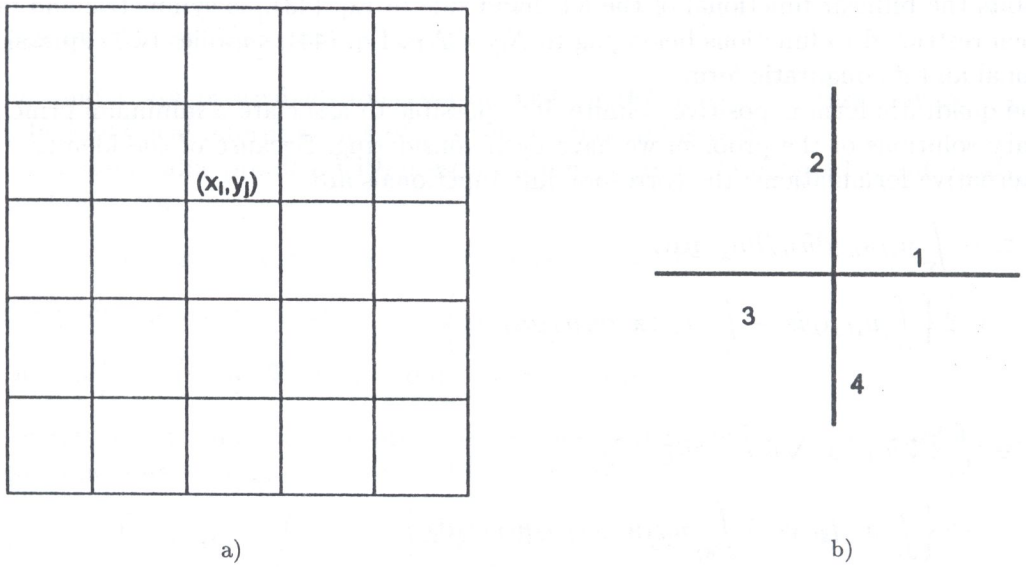


**Fig. 2.** a) Rectangular domain decomposition of $\Omega$; b) Numbering of internal boundaries

Given any subregion $\Omega_{ij}$, consider functions $w \in N_Q$ whose support is contained in the closure of $\Omega_{ij}$. Such functions satisfy $\mathcal{L}^* w = 0$, locally and vanish on $\partial\Omega_{ij}$ — hence, on $\partial\Omega$. It will be assumed, further, that any function fulfilling these conditions is determined by its trace on $\Sigma_{ij}$. With each subregion $\Omega_{ij}$, a system of weighting functions $\{w_{ij}^{\alpha};\ \alpha = 1, \ldots\}$, belonging to $N_Q$, and satisfying the above properties, will be associated. It will be assumed that the restrictions to $\Sigma$, of the functions $w_{ij}^{\alpha}$, span $L^2(\Sigma)$, when $\alpha = 1, \ldots$, $i = 0, 1, \ldots, E_x$, and $j = 0, \ldots, E_y$. A convenient way of constructing a system with this property is explained next.

Using the numbering of internal boundaries of Fig. 2b, associated with each node $(x_i, y_j)$ five groups of weighting functions will be constructed:

**Group 0** — This group is made of only one function, which is linear in each one of the four interior boundaries between the squares of Fig. 1b, and such that $w_{ij}(x_i, y_j) = 1$.

**Group 1** — The restricion to interval "1", of Fig. 1b, is a polynomial in $x$, which vanishes at the end points of interval "1". There is one such polynomial for each degree.

**Group 2** — The restricion to interval "2", of Fig. 1b, is a polynomial in $y$, which vanishes at the end points of interval "2". There is one such polynomial for each degree.

**Group 3** — The restricion to interval "3", of Fig. 1b, is a polynomial in $x$, which vanishes at the end points of interval "3". There is one such polynomial for each degree.

**Group 4** — The restricion to interval "4", of Fig. 1b, is a polynomial in $y$, which vanishes at the end points of interval "4". There is one such polynomial for each degree.

The support of the function of Group "0", is the whole square, while those associated with Groups "1 to "4", have as support rectangles which can be obtained from each other by rotation, as shown in Fig. 3.

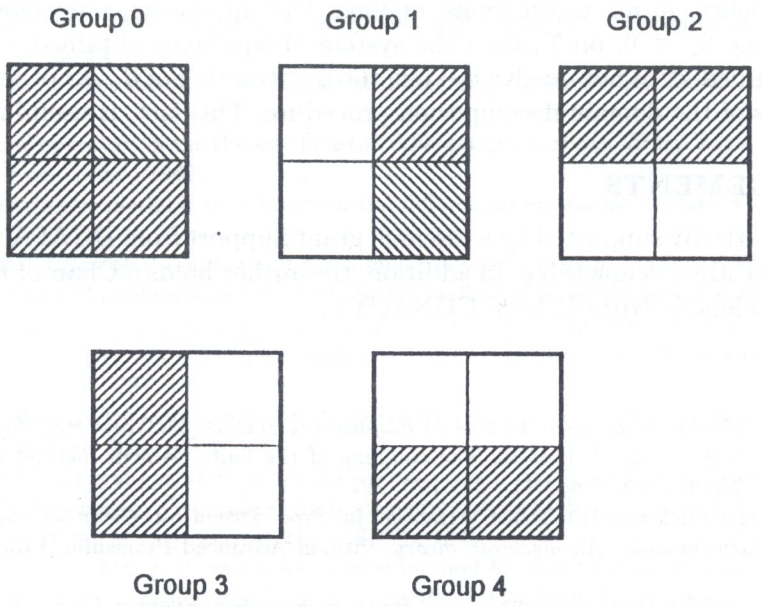


**Fig. 3.** The five groups of weighting functions, according to their supports

It must be observed that the above construction does not lead directly to a system of linearly independent functions. This is due to the fact that each couple of neighboring nodes, such as those limiting interval "1", share the polynomials defined on interval joining them. Thus, for example, what is what interval "1" for the node on the left, is interval "3", for the node on the right, and the corresponding polynomials are counted twice. Such repetition can be avoided if only polynomials of even degree are incorporated and, for each degree, the one associated with the left–node is linearly independent of that associated with the right–node. Even more, these two polynomials must be selected so that some linear combinations of them yield polynomials one degree lower.

Optimal trial functions $\phi \in D_1$, must satisfy the equation $\mathcal{L}\phi = 0$, and in actual implementations this is approximately fulfilled only, in most cases. Their construction is quite similar to that of the test functions. Actually, when the differential operator is symmetric, $\mathcal{L} = \mathcal{L}^*$, and trial and test functions can be chosen to be the same, at least for the Dirichlet boundary conditions here discussed.

## 7. Discretization

In actual applications the equations must be discretized. To achieve this it is necessary to restrict the degree of the polynomials, above, to be less or equal than certain number "$G$". When this is done, and the variational principle of Eq. (43) is applied, a system of equations is obtained. For 2-D problems, the structure of the matrix is block nine-diagonal, the blocks being $(2G-1) \times (2G-1)$.

In general, the test functions at each subregion are not known and must be constructed. A simple and convenient numerical procedure is by collocation. When this is applied, this leads to collocation method which is not standard. Numerical procedures based on the formulation here presented, require evaluation on the function only and the derivatives of the sought solution need not be evaluated. This is an advantage of TH–Collocation over "standard collocation" [75], for which it is necessary to solve for both, the function and its derivatives.

In the case when the differential operator is symmetric and positive definite, the matrix of the resulting system of equations, also enjoys this property, and direct application of conjugate gradient method is feasible.

The representation of the approximate solution is:

$$\widehat{u}(x,y) = \widehat{u}_P(x,y) + \sum_i \sum_j \sum_{\alpha}^{2G-1} U_{ij}^{\alpha} \phi_{ij}^{\alpha}(x,y) \ ,$$

where $\widehat{u}_P(x,y)$ satisfies, in an approximate manner, the inhomogeneous equation $\mathcal{L}\widehat{u}_P = f_\Omega$. If $\widehat{u}_P$ is chosen fulfilling $\widehat{u}_P \equiv 0$, on $\Sigma$, then the system of equations obtained when the variational equation (43), is applied, does not involve $\widehat{u}_P$; i.e., the construction of $\widehat{u}_P$ is not required to derive the system of equations of the domain decomposion procedure. This fact, is useful in some aplications.

## ACKNOWLDEGEMENTS

This research was partially supported by a research grant Support from DGAPA IN-102695, UNAM-CRAY SC005596 are also acknowledge. In addition, the author holds a Chair of Excellence (Catedra Patrimonial de Excelencia: Nivel I) from CONACYT.

## REFERENCES

[1] O.C. Zienkiewicz. *The Finite Element Method in Engineering Science.* McGraw–Hill, New York, 1977.

[2] O.C. Zienkiewicz, D.W. Kelly, P. Bettess. The coupling of the finite element method and boundary solution procedures. *Int. J. Num. Math. Eng.*, **11**: 355–375, 1977.

[3] E. Trefftz. Ein Gregenstück zum Ritz'schen Verfahren. In: *Proc. Second Int. Congress Appl. Mech.*, Zurich, 1926.

[4] I. Herrera. *Boundary methods. An algebraic theory.* Pitman Advanced Publishing Program, Boston–London–Melbourne, 1984.

[5] I. Herrera. Trefftz Method. In: C.A. Brebbia, ed., *Topics in Boundary Element Research*, Vol.1: *Basic Principles and Applications*, Springer–Verlag, Chapter 10, 225–253, 1984.

[6] I. Herrera. Boundary Methods for Fluids. In: R.H. Gallagher, D. Norrie, J.T. Oden, O.C. Zienkiewicz, eds., *Finite Elements in Fluids*, Chapter XIX del Vol. IV, 403–432, John Wiley & Sons Ltd., 1982.

[7] S.G. Mikhlin. *Variational methods in mathematical physics.* Pergamon Press, 1964.

[8] K. Recktorys. *Variational methods in mathematics, science and engineering.* D. Reidel Pub., Co., 1977.

[9] V.D. Kupradze et al. *Three dimensional problems of the mathematical theory of elasticity and thermoelasticity.* North Holland, 1979.

[10] L. Amerio. Sul calculo delle autosoluzioni dei problemi al contorno per le equazioni lineari del secondo ordine di tipo ellitico. *Rend. Acc. Lincei*, **1**: 352–359 and 505–509, 1946.

[11] G. Fichera. Teoremi di completezza sulla frontiera di un dominio per taluni sistema di funzioni. *Ann. Mat. Pura e Appl.*, **27**: 1–28, 1948.

[12] M. Picone. Nouvi methodi risolutivi per i problem d'integrazioni lineari a derivati parziali e nuova applicazione delle transformate multipla di Laplace nel caso delle equazioni a coeficienti constanti. *Atti Acc. Sc. Torino*, **76**: 413–426, 1940.

[13] V.D. Kupradze. On the approximate solution of problems in mathematical physics. *Russian Math. Surveys*, **22**: 58–107, 1967.

[14] I.N. Vekua. *New methods for solving elliptic equations.* North Holland, Amsterdam etc. and John Wiley, New York, XII, 358, 1967.

[15] D. Colton, W. Waltzlawek. Complete families of solutions to the heat equation and generalized heat equation in $R^n$. *Jour. Differential Equations* , **25**: 96–107, 1977.

[16] I. Herrera. Boundary methods. A criterion for completeness. *Proc. National Academy of Sciences, USA*, **77**: 4395–4398, 1980.

[17] F.J. Sánchez–Sesma, I. Herrera, H. Avilés. A boundary method for elastic wave diffraction. Application to scattering of SH–waves by surface irregularities. *Bull. Seismological Society of America.*, **72**: 473–490, 1982.

[18] I. Herrera. General variational principles applicable to the hybrid element method. *Proc. of the National Academy of Sciences, USA*, **74**: 2595–2597, 1977.

[19] I. Herrera. Theory of connectivity for formally symmetric operators. *Proc. National Academy of Sciences, USA*, **74**: 4722–4725, 1977.

[20] I. Herrera. Variational principles for problems with linear constraints. Prescribed jumps and continuation type restrictions. *J. Inst. of Mathematics and its Applics.*, **25**: 67–96, 1980.

[21] I. Herrera, J. Sabina. Connectivity as an alternative to boundary integral equations. Construction of bases. *Proc. National Academy of Sciences, USA*, **75**: 2059–2063, 1978.

[22] H. Gourgeon, I. Herrera. Boundary methods. C-complete systems for biharmonic equations. In: C.A. Brebbia, ed., *Boundary Element Methods*, 431–441, Springer Verlag, Berlin, 1981.

[23] I. Herrera, H. Gourgeon. Boundary methods C-complete systems for Stokes problems. *Computer Methods in Applied Mechanics & Engineering*, **30**: 225–241, 1982.

[24] I. Herrera, D.A. Spence. Framework for biorthogonal Fourier series. *Proc. National Academy of Sciences, USA*, **78**: 7240–7244, 1981.

[25] G. Alduncin, I. Herrera. Contribution to free boundary problems using boundary elements, Trefftz approach. *Computer Methods in Applied Mechanics and Engineering*, **42**: 257–271, 1984.

[26] A.P. Zielinski, I. Herrera. Trefftz method: Fitting boundary conditions. *International Journal for Numerical Methods in Engineering*, **24**: 871–891, 1987.

[27] R.P. Gilbert. Function theoretic methods in partial differential equations. *Academic Press, New York, XVIII*, **311**, 1969.

[28] S. Bergman. Integral operators in the theory of partial differential equations. *Erbeg. Math. Grenzgeb*, **23**, Spinger, Berlin etc., 1961; 2 rev. print., 1969.

[29] R.P. Gilbert. Constructive methods for elliptic partial differential equations. *Lecture Notes in Math.*, **365**, VII, 397, 1974.

[30] D.L. Colton. *Partial differential equations in the complex domain.* Pitman, London, 1976.

[31] D.L. Colton. *Solution of boundary value problems by the method of integral operators.* Pitman, London, 1976.

[32] D.L. Colton. *Analytic theory of partial differential equations.* Pitman, Boston etc., XII, 239, 1980.

[33] M. Kracht, E. Kreyszig. *Methods of complex analysis in partial differential equations with applications.* Wiley & Sons, New York etc., XIV, 394, 1988.

[34] H. Begehr, R.P. Gilbert. *Transformations, transmutations, and kernel functions*, Vol. 1. Longman Scientific & Technical Essex, England, 399, 1992.

[35] E. Lanckau. *Complex integral operators in mathematical physics.* Akademie–Verlag Berlin, (to appear).

[36] A.P. Zielinski, O.C. Zienkiewicz. Generalized finite element analysis with T-complete boundary solution function. *Int. Jour. for Num. Meths in Engineering*, **21**: 509–528, 1985.

[37] H.B. Li, G.M. Han, H.A. Mang, P. Torziecky. A new method for the coupling of finite element and boundary element discretized subdomains of elastic bodies. *Computer Methods in Applied Mechs and Engineering*, **54**: 2, 161–185, 1986.

[38] J. Jirousek, L. Guex. The hybrid–Trefftz finite element model and its application to plate bending. *Int. J. Num. for Num. Meths in Engineering*, **23**: 651–666, 1986.

[39] Z. Badics. A boundary matching method in solving 3–dimensional steady–state eddy–current problems. *IEEE Transactions on Magnetics*, **24**: 1, 142–145, 1988.

[40] F.J. Sánchez–Sesma, I. Herrera. Surface motion of topographical irregularities for incident P, Sv and Rayleigh Waves. *Bull. of the Seismol. Soc. of America*, **75**: 1, 263–269, 1985.

[41] H. Eshraghi, M. Dravinski. Transient scattering of elastic–waves by dipping layers of arbitrary shape. 2. Plane–Strain Model. *Earthquake Engineering & Structural Dynamics*, **18**: 3, 417–434, 1989.

[42] S. Hirose, M. Kitahara. Elastic Wave scattering by inhomogeneous and anisotropic bodes in a half–space. *Earthquake Engineering & Structural Dynamics*, **18**: 2, 285–297, 1989.

[43] Ch. Zhang, O.A. Pekau, F. Jin. Application of Fe–Be–Ibe coupling to dynamic interaction between alluvial soil and rock canyons. *Earthquake Eng. Struc. Dynam.*, **21**: 5, 367–385, 1992.

[44] F.J. Sánchez–Sesma, M. Campillo. Topographic effects for incident P, Sv and Raileigh–Waves. *Tectonophysics*, **218**: 113–125, 1993.

[45] I. Herrera. Unified approach to numerical methods. Part 1. Green's formulas for operators in discontinuous fields. *Numerical Methods for Partial Differential Equations*, **1**: 12–37, 1985.

[46] I. Herrera. Unified Approach to Numerical Methods. Part 2. Finite elements, boundary methods and its coupling. *Numerical Methods for Partial Differential Equations*, **1**: 159–186, 1985.

[47] I. Herrera, L. Chargoy, G. Alduncin. Unified approach to numerical methods. Part 3. Finite differences and ordinary differential equations. *Numerical Methods for Partial Differrential Equations*, **1**: 241–258, 1985.

[48] I. Herrera. Some unifying concepts in applied mathematics. In: R.E. Ewing, K.I. Gross, C.F. Martin, eds., *The Merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics*, Springer Verlag, New York, 1986.

[49] I. Herrera. Localized Adjoint Methods: A New Discretization Methodology. In: W.E. Fitzgibbon, M.F. Wheeler, eds., *Computational Methods in Geosciences*, Chapter 6, SIAM, Philadelphia, 1992.

[50] I. Herrera. The algebraic theory approach to ordinary differential equations: Highly accurate finite differences. In: *Numerical Methods for Partial Differential Equations*, John Wiley, New York, **3**: 3, 199–218, 1987.

[51] M.A. Celia, I. Herrera, E.T. Bouloutas. Adjoint Petrov–Galerkin Methods for Multi-Dimensional Flow Problems. In: T.J. Chung, R. Karr, eds., *Finite Element Analysis in Fluids*, University of Alabama Press, 953–958, 1989.

[52] M.A. Celia, I. Herrera, E.T. Bouloutas, J.S. Kindred. A new numerical approach for the advective diffusive transport equation. *Numer. Methods for Partial Differential Equations*, **5**: 3, 203–226, 1989.

[53] M.A. Celia, J. S. Kindred, I. Herrera. Contaminant Transport and Biodegradation. 1: A numerical model for reactive transport in pourous media. *Water Resour. Res.*, **25**: 6, 1141–1148, 1989.

[54] I. Herrera, R.E. Ewing, M.A. Celia, T.F. Russell. Eulerian–Lagrangian Localized Adjoint Method: The Theoretical Framework. *Numerical Methods for Partial Differential Equations*, **9**: 431–457, 1993.

[55] I. Herrera, M.A. Celia, T.F. Russell, R. Ewing. An Eulerian–Lagrangian Localized Adjoint Method for the Advection–Diffusion Equation. *Advances in Water Resources*, **13**: 4, 187–206, 1990.

[56] I. Herrera, R.E. Ewing. Localized Adjoint Methods: Applications to Multiphase Flow Problems. *Proceedings Fifth Wyoming Enhanced Oil Recovery Symposium*, May 10–11, 1989, Enhanced Oil Recovery Institute, University of Wyoming, 155–173, 1990.

[57] M.A. Celia, S. Zisman. An Eulerian–Lagrangian localized adjoint method for reactive transport in groundwater. In: *Computational Methods in Subsurface Hydrology, Proceedings of the Eight International Conference on Computational Methods in Water Resources*, Venice, Italy, 383, 1990.

[58] R.E. Ewing, H. Wang. Eulerian–Lagrangian localized adjoint methods for reactive transport in groundwater. In: Wheeler, et al., eds., *Environmental Studies, IMA Volume, Mathematics and Its Applications*, Springer Verlag, Berlin (to appear).

[59] G. Herrera, I. Herrera. An Eulerian–Lagrangian Method of Cells, Based on Localized Adjoint Methods. *Numerical Methods for Partial Differential Equations*, **10**: 205–223, 1994.

[60] H.K. Dhale, R.E. Ewing, T.F. Russell. Eulerian–Lagrangian localized adjoint methods for a nonlinear convection–diffusion equation. *Comp. Meth. Appl. Mech. Engrg.*, (accepted).

[61] R.E. Ewing, H. Wang. Eulerian–Lagrangian localilzed adjoint methods for linear advection or advection–reaction equations, and their convergence analysis. *Comp. Mech.*, **12**: 97, 1993.

[62] H. Wang, R.E. Ewing. Optimal–Order Convergence Rates for Eulerian–Lagrangian localized Adjoint Methods for Reactive Transport and Contamination in Groundwater. *Numerical Methods for Partial Differential Equations*, **11**: 1, 1995.

[63] M.A. Celia. Eulerian–Lagrangian localized adjoint methods for contaminant transport simulations. *Computational Methods in Water Resources X*, **1**: 207–216, 1994.

[64] I. Herrera. Innovative Discretization Methodologies Based on LAM. In: K. Morgan et al., eds., *Finite Elements in Fluids: New Trends and Applications*, **2**, Pineridge Press, 1437–1447, 1993.

[65] D.E. Keyes, W.D. Gropp. A comparison of Domain Decomposition techniques for elliptic partial differential equations and their parallel implementation. *SIAM J. Sci. Stat. Comput*, **8**: 166–201, 1987.

[66] M. Israeli, L. Vozovoi. Domain decomposition methods for solving parabolic PDEs on multiprocessors. *Applied Numerical Mathematics*, **12**: 193–212, 1993.

[67] R. Glowinski, G.H. Golub, G.A. Meurant, J. Périaux. *Domain Decomposition methods for partial differential equations*, Vol. 1, Ecole Nationale des ponts et Chaussees, Paris, France, January, 1987. *SIAM*, Philadelphia, 1988.

[68] T.F. Chan, R. Glowinsky, J. Périaux, O.B. Widlund. *Domain decomposition methods*, Vol. 2. Los Angeles, CA., January 14–16, 1988 *SIAM*, Philadelphia, 1989.

[69] T.F. Chan, R. Glowinsky, J. Périaux, O.B. Widlund. *Domain decomposition methods for partial differential equations*, Vol. 3. *SIAM*, Philadelphia, 1989.

[70] R. Glowinski, Y.A. Kuznetsov, G. Meurant, J. Périaux, O.F. Widlund. *Domain decomposition methods for partial differential equations*, Vol. 4, Moscow, May 21–25, 1990 *SIAM*, Philadelphia, 1990.

[71] D.E. Keyes, T.F. Chan, G. Meurant, J.S. Scroggs, R.G. Voigt. *Domain decomposition methods for partial differential equations*, Vol. 5, Norfolk, VA., May 6–8, 1991 *SIAM*, Philadelphia, 1991.

[72] A. Quarteroni, J. Periaux, Y.A. Kuznetsov, O.F. Widlund. *Domain decomposition methods in science and engineering*, Vol. 6 Am. Math. Soc., Providence, R.I., 1992.

[73] D.E. Keyes, J. Xu. *Domain decomposition methods in scientific and engineering computing*, Vol. 7, Am. Math. Soc., Providence, R.I., 1993.

[74] I. Herrera. Trefftz–Herrera domain decomposition. *Advances in Engineering Software*, **24**: 43–56, 1995.

[75] M.A. Celia. Collocation on deformed finite elements and alternating direction collocation methods. Doctoral dissertation, Department of Civil Engineering, Princeton Unicersity, September 1983.

[76] I. Herrera. On operator extensions: The Algebraic theory approach. *Advances in Optimization and Numerical Analysis, Proc. of 6th Workshop on Optimization and Numerical Analysis*, Oaxaca, México. Kluwer Academic Publishers, 155–163, 1992.

[77] B.M. Allen, I. Herrera, G.F. Pinder. *Numerical Modeling in Science and Engineering*, John Wiley, New York, 418, 1988.