

Trefftz type approximation and the generalized finite element method — history and development

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This paper shows the essential unity between the known approximation procedures to differential equations. The boundary type approximations associated with Trefftz are shown to be a particular form of the weighted residual approximation and can provide a very useful basis for generating hybrid 'finite elements'. These can be used in combination with other finite element approaches.

1. INTRODUCTION

In his 1926 paper Trefftz introduced a general approximation procedure for the solution of self adjoint differential equations [1] which could compete with finite differences or with Raleigh–Ritz type methods. In the author's view [2, 3] this was a forerunner of all boundary type solution methods (occasionally and inconsistently referred to as boundary element methods). Though in his original work the basis functions were simply sets of continuous functions satisfying the homogeneous governing equations of the problem, an alternative formulation is possible. In this the solution is presented as a boundary integral using suitable, fundamental, solutions (or "Green's functions") of the differential equation. By using such fundamental solutions the exact solution of the original problem can be recast as that involving the solution of an integral equation with singular kernels. It is of interest to observe that much of the currently used boundary solution process evolved from such forms [4–11]. However, an early attempt to use such integral equation solutions was made also in 1926 by R. Miche [12] and was published incidentally in the same proceedings as the original paper by Trefftz.

The numerical solution of the integral equations is generally attempted by using some form of collocation. This results in two major difficulties. First, the integrals over singular kernels involve complex operations if accuracy is to be maintained. Second, the resulting algebraic equation systems are not symmetric, involving increased computational costs and prohibiting a simple coupling with finite element schemes.

The symmetry can be re-established as shown by the present author [2, 3] but involves a double integration and hence, obviously, much larger computational costs.

For obvious reasons it appears to many that an approach in a manner outlined by Trefftz which does not involve singular integrals would be advantageous. Here symmetry can be readily achieved, as will be shown later, and the difficult integration problems disappear. Early attempts at such solutions had been made by Quinlan [13, 14] and later by Jirousek [15–18] who introduced a very effective concept of employing the exact solution within a frame which could readily be linked to standard finite elements.

The "hybrid" development can provide rapidly convergent elements and promises to be useful. Others have enlarged on the procedures here suggested [19] and Herrera provided much of the background necessary for the choice of suitable basis functions [20, 21]. The general methodology

can well be now named the Trefftz method and this name has now been widely adopted [22, 23]. Indeed, the development of elements based on the Trefftz approximation by Jirousek et al. enabled the boundary solution methods to be readily coupled with standard finite element methods. This objective was one which prompted the work reported in [2, 3] by the author and indeed a similar approach by Stein [24].

In the next section we shall show how the Trefftz method fits into a general form of approximation used in finite element process and indeed how Trefftz type elements can be developed.

2. A UNIFIED APPROXIMATION PROCESS AND THE TREFFTZ METHOD (GENERALISED FINITE ELEMENT METHODS)

It is evident, at least to this author, that all known approximation methods used in the solution of self adjoint, partial differential equations, have a common ancestor — *the weighted residual method*. As the standard finite element processes today are very widely used, we could alternatively adopt the nomenclature of *generalised finite element method* to describe the unified procedure [25] but this is, of course, a matter of personal preference.

The unified procedure follows the steps given below:

- (i) *The problem*. This is defined as a solution of a partial differential equation system.

$$\mathbf{A}(\mathbf{u}) \equiv \mathbf{L}\mathbf{u} + \mathbf{p} = \mathbf{0} \quad \text{in domain } \Omega, \quad (1)$$

where the operator \mathbf{L} is of the form

$$\mathbf{L} \equiv \mathbf{S}^T \mathbf{D} \mathbf{S} \quad (2)$$

and \mathbf{D} is a symmetric, positively definite matrix. The boundary conditions are written as

$$\mathbf{B}\mathbf{u} \equiv \mathbf{M}\mathbf{u} + \mathbf{q} \quad \text{on } \Gamma \equiv \partial\Omega. \quad (3)$$

- (ii) A *trial* solution is assumed in the form

$$\hat{\mathbf{u}} = \mathbf{N}\mathbf{a}, \quad (4)$$

where \mathbf{a} is a set of unknown parameters

$$\mathbf{a} = [a_1, \dots, a_n] \quad (5)$$

and \mathbf{N} is the set of *basis* or *shape* functions corresponding to these parameters.

$$\mathbf{N} = [N_1, \dots, N_n]. \quad (6)$$

- (iii) A *weighted residual* form is used to obtain the algebraic equations from which the parameters are obtained. Inserting $\hat{\mathbf{u}}$ of equation (4) into equations (1) and (3), the domain and boundary residuals are obtained and using a weighting function \mathbf{W} in Ω and $\bar{\mathbf{W}}$ on the boundaries, we can write

$$\int_{\Omega} \mathbf{W}^T [\mathbf{L}(\mathbf{N}\mathbf{a}) + \mathbf{p}] d\Omega + \int_{\Gamma} \bar{\mathbf{W}}^T [\mathbf{M}(\mathbf{N}\mathbf{a}) + \mathbf{q}] d\Gamma = \mathbf{0}. \quad (7)$$

As the number of algebraic equations has to correspond to the number of unknown parameters n , \mathbf{W} is of similar form to that given by equation (6).

$$\mathbf{W} = [W_1, W_2, \dots, W_n]. \quad (8)$$

The above algebraic equations can always be written as

$$\mathbf{K}\mathbf{a} = \mathbf{f}, \quad (9)$$

here, \mathbf{K} has finite element properties:

$$\mathbf{K}_\Omega = \sum \mathbf{K}_{\Omega^e}. \quad (10)$$

The various well known approximation procedures are simply derived using various types of weighting and trial functions. Thus we can see that we obtain:

'Standard' FEM when

- (a) $\mathbf{W} = \mathbf{N} = -\bar{\mathbf{W}}$ (Galerkin),
 (b) \mathbf{N} defined on local basis (small support) \Rightarrow narrow band equations. Symmetric \mathbf{K} is obtained by integration by parts which reduces continuity requirements for self adjoint problems.

'Spectral' Methods

As above but \mathbf{N} defined by series of trigonometric or hierarchical polynomials \Rightarrow wide band equations, etc. as above.

Finite Difference Method

- (a) $\mathbf{W}_i = \delta_i(\mathbf{a}_i)$ δ is Dirac function.
 (b) \mathbf{N} is often discontinuous and needs higher differentiability.

This is a *Collocation* process.

Finite Volume Method

- $\mathbf{W}_i = \mathbf{I}$ in Ω_i ,
 $\mathbf{W}_i = \mathbf{0}$ elsewhere.

Subdomain collocation is recognised here.

The Trefftz Method

- (a) \mathbf{N}_i chosen so that homogeneous part of the governing equation is satisfied i.e:

$$\mathbf{L}\mathbf{N}_i \equiv \mathbf{0}. \quad (11)$$

- (b) Approximation equations remain only on boundary.

In self adjoint problems (eq. solid mechanics) these are the form:

$$\int_{\Gamma_t} \mathbf{W}_t^T(\mathbf{t} - \bar{\mathbf{t}}) d\Gamma + \int_{\Omega_u} \mathbf{W}_u^T(\mathbf{u} - \bar{\mathbf{u}}) d\Omega = 0, \quad (12)$$

where:

$\bar{\mathbf{u}}$ = prescribed \mathbf{u} on Γ_u ,

$\bar{\mathbf{t}}$ = prescribed \mathbf{t} on Γ_t .

and

$$\Gamma_u \cup \Gamma_t = \Gamma.$$

Note that we can write both the 'stresses' and the 'tractions' in terms of the function \mathbf{u} as:

$$\mathbf{u} = \mathbf{N}\mathbf{a}, \quad \boldsymbol{\sigma} = \mathbf{D}\mathbf{S}\mathbf{u}, \quad \mathbf{t} = \mathbf{G}\boldsymbol{\sigma}. \quad (13)$$

Various choices for \mathbf{W}_t and \mathbf{W}_u are possible and have been used in various applications together with:

- Collocation point or subdomain (non-symmetric \mathbf{K}),
- Least squares (symmetric \mathbf{K}),
- GALERKIN-VARIATIONAL (symmetric \mathbf{K}).

Clearly the Trefftz method results in a reduction of the number of equations as these have to be formed only on the boundaries. It is, however, obvious now that all the algebraic equations are coupled and matrices are full as the functions \mathbf{N} will need to be defined over the whole domain.

In many boundary solutions processes the functions \mathbf{N} involve integrals of the "fundamental" solution. However, if non singular approximations are used the most convenient choice of the weighting functions is the Galerkin variational procedure. We can write

$$\mathbf{W}_t = \delta \mathbf{U} = \mathbf{N}, \quad (14)$$

$$\mathbf{W}_u = \delta \mathbf{t} = \mathbf{GD}(\mathbf{SN}) \quad \text{with} \quad \delta \mathbf{a} \equiv \mathbf{I} \quad (15)$$

and the approximation equations are:

$$\int_{\Gamma_t} \delta \mathbf{u}^T (\mathbf{t} - \bar{\mathbf{t}}) d\Gamma - \int_{\Gamma_u} \delta \mathbf{t}^T (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma = 0 \quad (16)$$

↓

$$\mathbf{K} \mathbf{a} = \mathbf{f}, \quad (17)$$

where

$$\mathbf{K} = \int_{\Gamma_u} [\mathbf{GD}(\mathbf{SN})]^T \mathbf{N} d\Gamma - \int_{\Gamma_t} \mathbf{N}^T \mathbf{GD}(\mathbf{SN}) d\Gamma. \quad (18)$$

This matrix is always symmetric as:

$$\int_{\Gamma} \mathbf{t}^T \mathbf{u} d\Gamma \equiv \int_{\Gamma} \mathbf{u}^T \mathbf{t} d\Gamma \equiv \text{Energy}. \quad (19)$$

The approximation contained in Equation 16 permits a solution to be obtained for any domain on the boundaries of which tractions or displacements are prescribed.

In the formulation we have excluded the body forces, on non homogeneous terms arising in the governing equations (viz \mathbf{p}). These can obviously be included with no difficulty.

3. SUPER ELEMENTS AND LINKS WITH STANDARD FINITE ELEMENTS

It is of course possible to link domains in which a particular Trefftz style approximation is used with others in which a different type of Trefftz approximation is employed or indeed with a standard type of finite element. The possible objectives of this have been described quite early, [2, 3 and 24]. One of these is obviously the exploitation of the ease with which Trefftz approximations can deal with infinite or singular domains while retaining the versatility of standard elements in dealing with nonlinearities. Various procedures for linking can and have been used. However, the most direct and simple is the use of the "frame" concepts [2,15 and 18,25] leading to forms which are similar to the hybrid finite element formulation, viz. Fig. 1.

In this we consider that on the boundaries of the Trefftz domain discussed in the previous section the values of \mathbf{u} prescribed on Γ_u can be interpolated from nodal values \mathbf{V} as:

$$\tilde{\mathbf{u}} = \tilde{\mathbf{N}} \mathbf{V}. \quad (20)$$

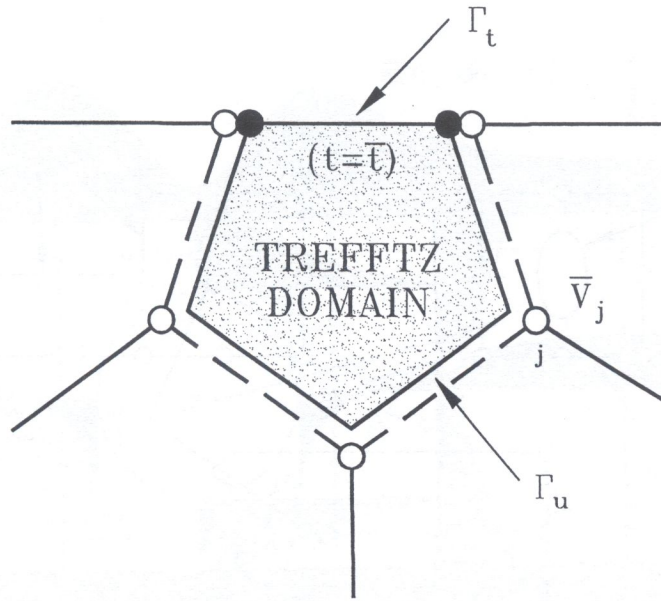


Fig. 1. A typical Trefftz-type element with \mathbf{u} prescribed on the frame Γ_u

We note that the corresponding “forces” transferred to outside of the frame are

$$\mathbf{q} = \int_{\Gamma_u} \tilde{\mathbf{N}}^T \mathbf{t} \, d\Gamma = \mathbf{Q}^T \mathbf{a}, \quad (21)$$

where

$$\mathbf{Q}^T = \int \tilde{\mathbf{N}}^T \mathbf{G} \mathbf{D}(\mathbf{S}\mathbf{N}) \, d\Gamma \quad (22)$$

and we can write combining Eqs. 16, 17, 20, 21 and 22

$$\begin{bmatrix} \mathbf{K} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{q} \end{Bmatrix}. \quad (23)$$

The parameters \mathbf{a} can be eliminated being defined strictly in the interior and a standard stiffness matrix form is obtained for the superelement

$$\mathbf{H}\mathbf{v} = \mathbf{q}. \quad (24)$$

Many usual and unusual forms are possible e.g. Infinite Elements, including corners, singularities, etc. as shown in Fig. 2

Trefftz type “elements” combined with standard finite elements allow the boundaries to take various shape as shown in Fig. 3. Convergence studies reported by Jirousek [18] and many others show that usually convergence to the exact solution is achieved better by using large elements of Trefftz type rather than a fine sub-division. This need not, however, always be so and Jirousek [26] has shown how a very simple, triangular element with 9 degrees of freedom can be developed from a hybrid Trefftz basis and that this can compete as a “small element” directly with others derived by conventional assumptions. Therefore, conclusions about the size and nature of the Trefftz type elements are somewhat problem dependent. In Fig. 4 we show some results of an analysis carried out by Jirousek [25] in which subdivision using standard and Trefftz type elements is used.

The subdivision of a Trefftz type domain is occasionally necessitated by the fact that the Trefftz solutions are only available for homogeneous regions. In Fig. 5 we show how the linking of two domains with different material characteristics can be obtained.

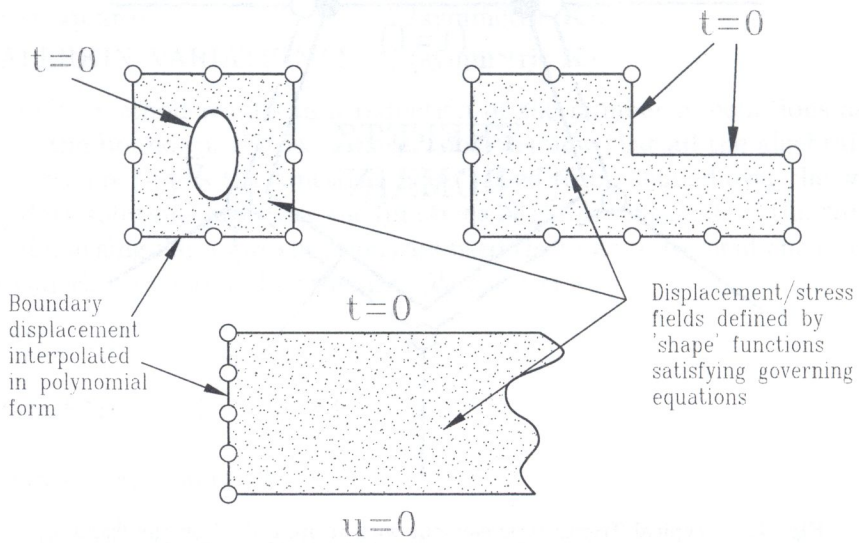


Fig. 2. Boundary-Trefftz-type elements. Some useful general forms

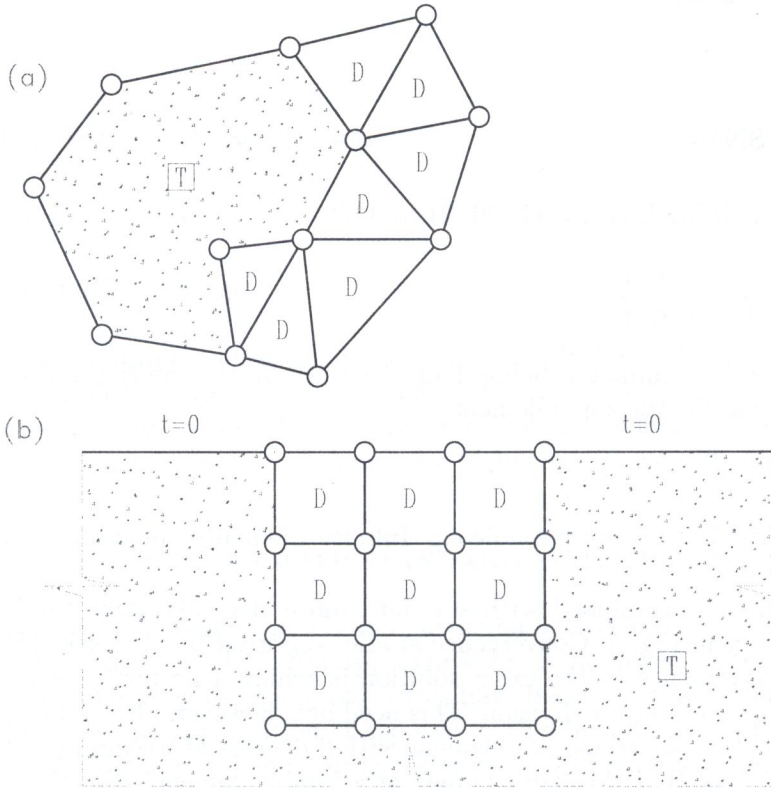
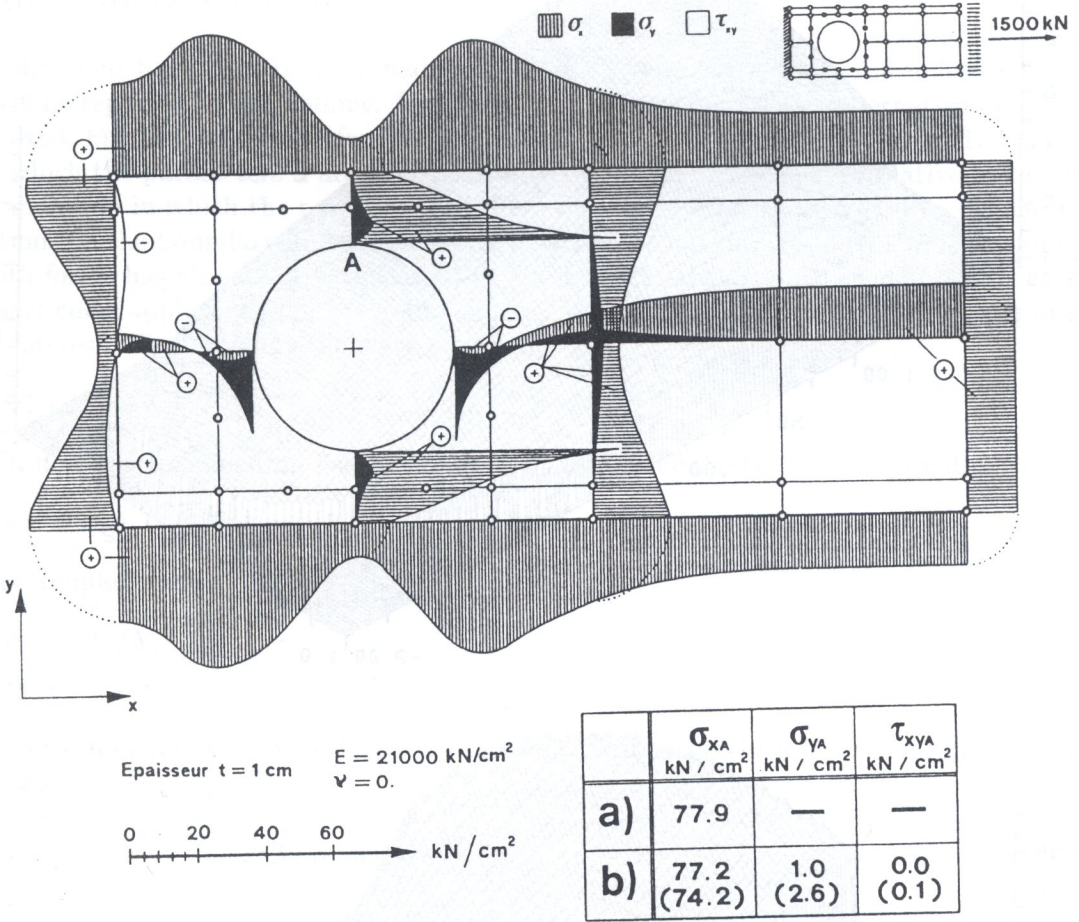


Fig. 3. Boundary-Trefftz-type elements (T) with complex-shaped 'frames' allowing combination with standard, displacement elements (D): (a) an interior element; (b) an exterior element

(a)



(b)

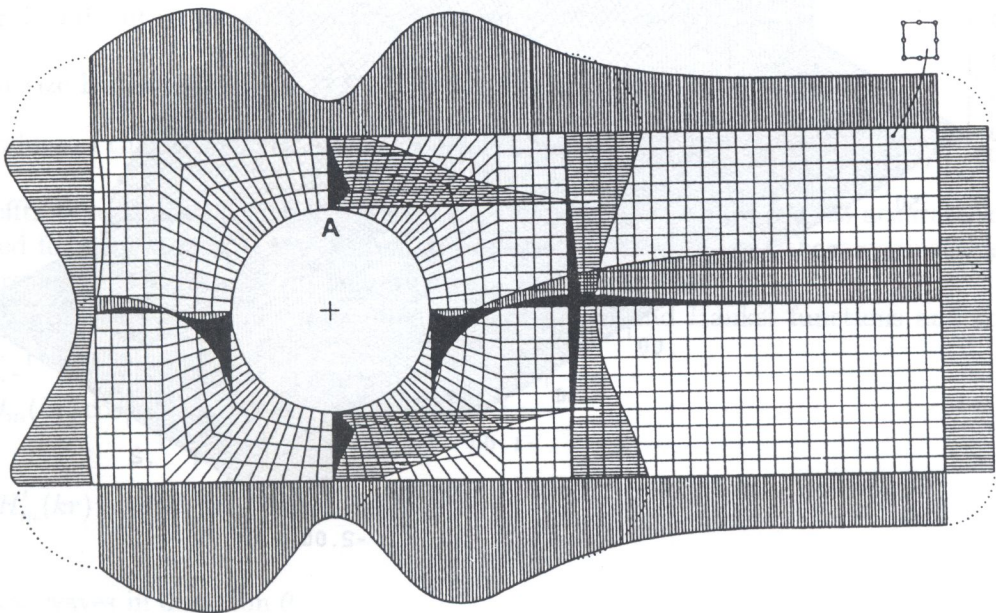


Fig. 4. Application of Trefftz-type elements to a problem of plane-stress tension bar with circular hole. (a) Trefftz element solution. (b) Standard displacement element solution. (Numbers in parenthesis indicate standard solution with 230 elements, 1600 DOF).

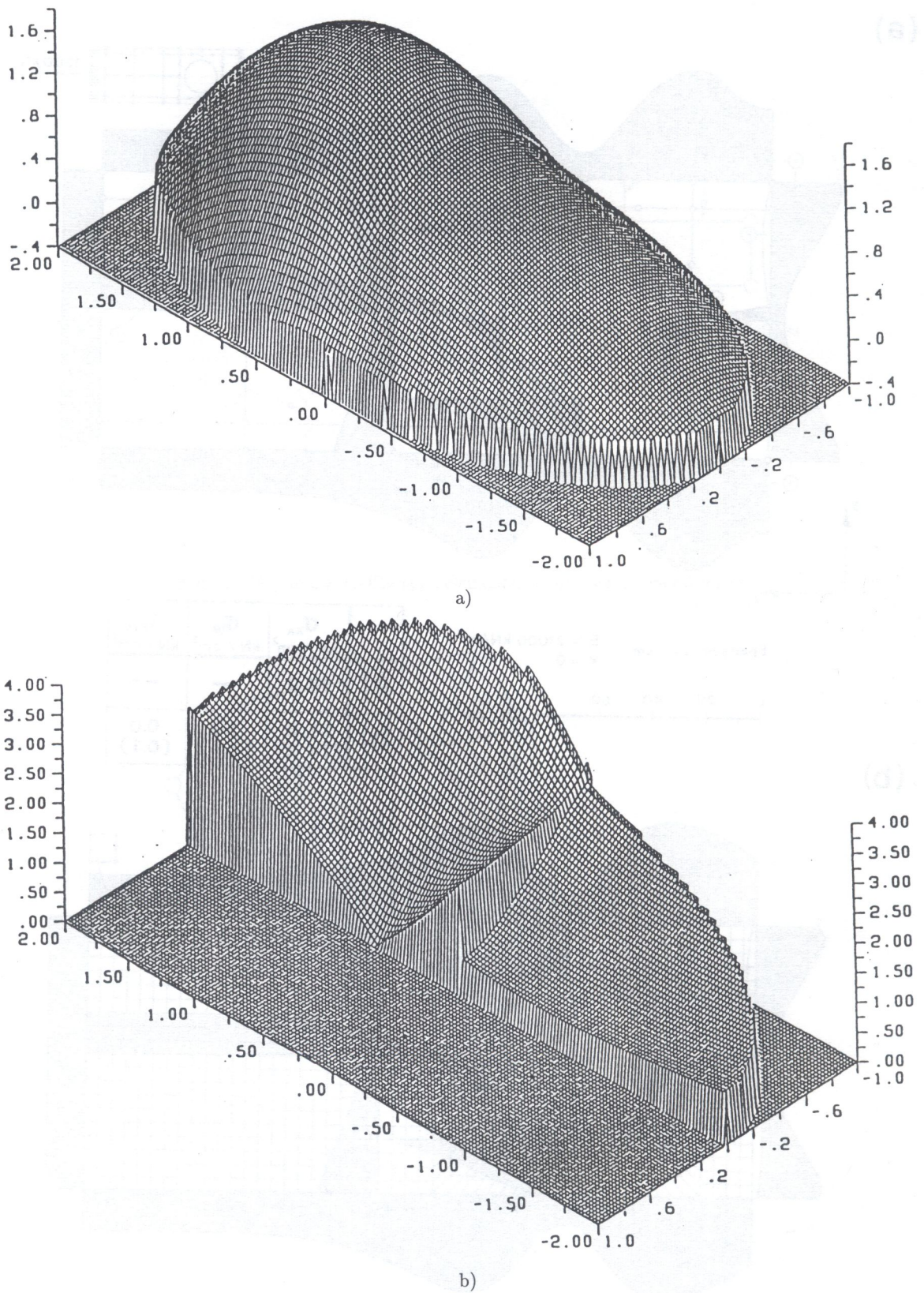


Fig. 5. Boundary-Treftz-type 'elements' linking two domains of different materials in an elliptic bar subject to torsion (Poisson equation). (a) Stress function given by internal variables showing almost complete continuity. (b) x component of shear stress (gradient of stress function showing abrupt discontinuity of material junction).

4. SOME APPLICATIONS OF TREFFTZ METHODS. THE HELMHOLTZ PROBLEM

Many applications have already been made of the Trefftz type of solutions and indeed further examples will be reported in this volume. As explained previously, numerous alternatives for creating elements exist. Further, we have so far introduced the reader to the so called indirect Trefftz formulation in which the parameters \mathbf{a} are defined throughout the domain. An alternative to this is the direct formulation in which the parameters of the problem are directly given as the displacements and tractions on the boundary. In the Appendix we show the alternatives on a simple example and use of both forms has already been discussed in references 22 and 23.

One particular application of the Trefftz process, or indeed of the Trefftz elements, is of great practical interest. This concerns the wave equation

$$\nabla(c\nabla\Phi) + b\Phi = 0 \quad (25)$$

which in a homogeneous medium becomes the well known Helmholtz Acoustic equation:

$$\nabla^2\Phi + k^2\Phi = 0 \quad (26)$$

when Φ is complex and

$$k = \omega/c = 2\pi/\lambda,$$

$$\omega = 2\pi/T$$

($2\pi/\lambda$ — wave number, λ — wave length). Boundary conditions which have to be imposed in the problem are

$$\begin{aligned} \frac{\partial\Phi}{\partial n} &= 0 && \text{on solid boundary,} \\ \frac{\partial\Phi}{\partial n} + ik\Phi &= 0 && \text{on } \Gamma_\infty \text{ (radiation condition).} \end{aligned}$$

The problem is difficult to tackle effectively with standard finite elements as:

- (i) element size \mathbf{h} has to be small ($1/5$ to $1/20$) of the wavelength λ and
- (ii) large values of the wavelength number k lead to instability.

With Trefftz type approaches, above restrictions do not apply and in Ref. 23 accurate solutions were obtained for several simple single domain problems of interest to offshore engineering where wave pressures on structures need to be evaluated, viz. Fig. 6.

The Trefftz functions used in such problems were Bessel and Hankel functions and these are given below:

$$J_0(kr), J_m(kr) \cos m\theta, J_m(kr) \sin m\theta$$

or

$$H'_m(kr), H'_m(kr) \cos m\theta, H'_m(kr) \sin m\theta$$

or

set of plane waves in direction θ ,

where J, H are BESSEL/HANKEL functions.

The use of Trefftz type elements in this context is being investigated by the author. In particular the possibility of studying non homogeneous domains is being initiated as such problems are of interest in harbour oscillations when the depth is variable.

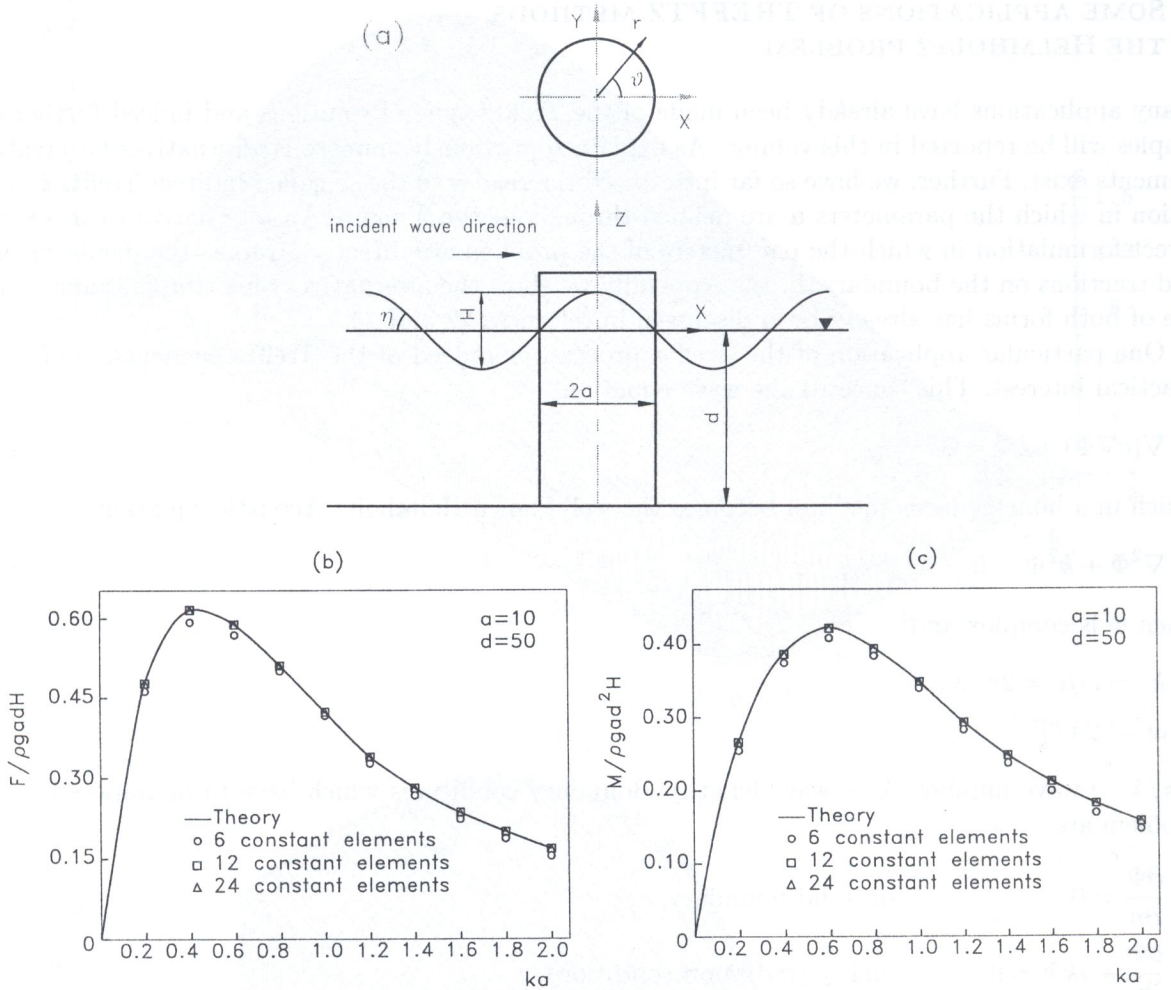


Fig. 6. Waves impacting on circular tower. (a) The problem. (b) Horizontal force. (c) Overturning moment computed for different values of ka .

5. CONCLUDING REMARKS

The possibilities offered by the application of Trefftz procedures are of interest and it seems without doubt that in the future Trefftz type elements will frequently be encountered in general finite element codes. We have not discussed here the problems of convergence and the relation between the number of unknown used in defining the Trefftz functions inside element domain and the number of unknowns defining the displacement, say, along the boundary. It can easily be shown that the number of unknowns within the element should be greater than that at the boundary. However, it should be noted that an increase of these interior unknowns will not lead to further improvement of convergence if it is increased beyond the necessary minimum.

It is the author's belief that the simple Trefftz approach will in the future displace much of the boundary type analysis with singular kernels.

APPENDIX

Direct and indirect procedures

For simplicity a specific example of Laplace equation has been chosen:

$$\nabla^2 u = 0 \quad \text{in } \Omega, \tag{A1}$$

$$u = \bar{u} \quad \text{on } \Gamma_u, \quad (\text{A2})$$

$$t = \frac{\partial u}{\partial n} \equiv \bar{t} \quad \text{on } \Gamma_t. \quad (\text{A3})$$

Weak form can be cast as

$$\int_{\Omega} W \nabla^2 u \, d\Omega + \int_{\Gamma_u} \frac{\partial W}{\partial n} (u - \bar{u}) \, d\Gamma - \int_{\Gamma_t} W \left(\frac{\partial u}{\partial n} - \bar{t} \right) \, d\Gamma = 0. \quad (\text{A4})$$

Two integrations by parts give:

$$\int_{\Omega} W \nabla^2 u \, d\Omega - \int_{\Omega} u \nabla^2 W \, d\Omega \equiv \int_{\Gamma} W \frac{\partial u}{\partial n} \, d\Gamma - \int_{\Gamma} u \frac{\partial W}{\partial n} \, d\Gamma \quad (\text{A5})$$

with

$$\begin{aligned} u &= \bar{u} \quad \text{on } \Gamma_u, \\ t &= \frac{\partial \bar{u}}{\partial n} \quad \text{on } \Gamma_t. \end{aligned} \quad (\text{A6})$$

Indirect solution

In this, as already discussed in the text, the trial solution is taken as

$$u \approx \hat{u} = \sum N_i a_i. \quad (\text{A7})$$

This does not identify physical variables and seeks a parameter set $\{a_i\}$ with

$$\nabla^2 N_i = 0. \quad (\text{A8})$$

If $W_j = N_j$ symmetric equations are obtained as already shown starting from equation (A5).

Direct solution

Starting from equation (A5) with:

$$W_j = N_j \quad \text{we have} \quad \nabla^2 W_j = 0. \quad (\text{A9})$$

and

$$\int_{\Gamma} N_j \frac{\partial u}{\partial n} \, d\Gamma - \int_{\Gamma} \frac{\partial N_j}{\partial n} u \, d\Gamma = 0.$$

If

$$\begin{aligned} u &= \bar{u} \quad \text{on } \Gamma_u, \\ \frac{\partial u}{\partial n} &= \bar{t} \quad \text{on } \Gamma_t. \end{aligned} \quad (\text{A10})$$

we approximate directly

$$\begin{aligned} t &= N_t \bar{t} \quad \text{on } \Gamma_u, \\ u &= N_u \bar{u} \quad \text{on } \Gamma_t \end{aligned} \quad (\text{A11})$$

and have an equation system

$$\mathbf{H}\mathbf{x} = \mathbf{f},$$

$$\mathbf{H} = \int_{\Gamma_u} \mathbf{N}^T \mathbf{N}_t \, d\Gamma - \int_{\Gamma_u} \left(\frac{\partial \mathbf{N}}{\partial n} \right)^T \mathbf{N}_u \, d\Gamma,$$

$$\mathbf{x} = \begin{Bmatrix} \bar{u} \\ \bar{t} \end{Bmatrix}, \quad (\text{A12})$$

which is non symmetric. Similar basis is frequently used with integral equation forms.

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