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# Analysis of rectangular thin plates and plate structures basing on the Vlasov's variational procedure[1]

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The solution procedure proposed by Vlasov based on the reduction of the basic two-dimensional boundary value problems into ordinary differential equations provides a good accuracy in the case of rectangular domains with small size ratios. The paper presents an extension of this method applied to rectangular Kirchhoff's plates in connection with the iterational scheme. The results are compared with analytical solutions available for rectangular plates with simplified boundary conditions and loading. The possibilities of application of the solutions for simple plate geometry to complex plate problems (e.g. complex geometry, boundary conditions) are discussed and illustrated by numerical examples.

## 1. Introduction

The elastic deformation model of a thin plate or shell is usually formulated as a two-dimensional boundary value problem. Analytical and numerical solution procedures are discussed and summarised in many monographes and textbooks, e.g. [7, 9, 12, 16]. Widely used numerical techniques are based on the variational or energy methods of the Rayleigh, Galerkin or Ritz–Timoshenko type. These methods specify the solution as a superposition of the shape functions with constant coefficients and lead to sets of linear algebraic equations. For simplified problems the solutions are represented in the form of series, whereas the shape functions are specified corresponding to the geometry of the whole domain, boundary conditions and the type of applied loads as trigonometrical functions, beam eigenfunctions, polynoms etc. In the case of complex geometries the finite element [18] or finite strip [8] discretization is used, whereas the shape functions are defined on the finite subdomains.

The variational method proposed by Vlasov [17] and Kantorovich [11] deals with determination of functions rather then constant coefficients and leads to the solution of ordinary matrix differential equations. In the case of rectangular domains the principal unknowns of the problem (e.g. displacements) are approximated by two sets of functions as follows

$$U(x,y) = \sum_{i=1}^{n} U_i(y)\chi_i(x), \tag{1}$$

[1]Dedicated to our colleague and friend Prof. Dr.-Ing. habil. Hans Göldner on the occasion of his seventieth birthday.

where $x$ and $y$ denote cartesian coordinates, $\chi_i(x)$ are given dimensionless displacements which must be determined for the given boundary conditions and load types for the coordinate direction $x$ and $U_i(y)$ are unknown generalized displacements. The Vlasov–Kantorovich variational procedure leads to sets of ordinary differential equations and associated boundary conditions which can be represented as follows

$$\sum_{j=1}^{n} L_{ij}[U_j(y)] = q_i(y), \quad \sum_{j=1}^{n} L_{kj}^{\Gamma}[U_j(y_\Gamma)] = B_k, \quad i = 1, \ldots, n, \tag{2}$$

where $L_{ij}$ and $L_{kj}^{\Gamma}$ are differential operators, $q_i(y)$ are generalized loads in the $y$ direction and $B_k, k = 1, 2, \ldots, m$ are given generalized displacements and (or) forces on the boundaries $y = 0$ and $y = l_y$, with $l_y$ as the in-plane size of the structure in the $y$ — direction. For different problems of thin-walled structures such as shells of moderate length, plates, folded structures, etc. solutions can be obtained even by use of the first series term in Eq. (1), cf. [1, 2, 3, 10, 17], whereas the function $\chi_1(x) \equiv \chi(x)$ is the unit displacement state of a beam or systems of beams. In this case the set of Eqs. (2) reduces to a single differential equation with constant coefficients and it's integration can be performed through analytical methods. However, if the beam functions are used, a good accuracy of the solution can be reached only if the ratio of the in–plane sizes of the shell or plate is sufficiently small. Here we use the Vlasov's procedure in connection with the iterational scheme. This allows to obtain the solution independent of the type of the given function $\chi(x)$. Both the functions $\chi(x)$ and $U_1(y) \equiv U(y)$ are defined as analytical solutions of the differential Eqs. (2) with constant coefficients obtained through numerical integrations.

The numerical–analytical solution and its accuracy are illustrated on examples of rectangular plates. Finally the application of such a solution for complex plate problems (e.g. in the case of complex plate geometry) is discussed.

## 2. Variational procedure

The classical plate theory which is based on the Kirchhoff's hypotheses leads to the solution of the Lagrange–Sophie Germain equation in cartesian coordinates

$$\nabla\nabla w = \frac{q(x,y)}{D}, \quad \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{3}$$

where $w(x,y)$ is the deflection of the middle surface, $q(x,y)$ is the applied load, $E$ is the Young's modulus and $\nu$ is the Poisson's ratio.

Firstly we consider a thin rectangular plate related to the dimensionless in–plane coordinates $\xi = x/l_x$, $\eta = y/l_y$, where $l_x$ and $l_y$ are in-plane plate sizes, Fig. 1. For the following derivations we assume that the applied load can be multiplicatively decomposed into two loads which depend only on a single coordinate $q(\xi, \eta) = q_\xi(\xi)q_\eta(\eta)$. According to the Vlasov's method the deflection function $w(\xi, \eta)$ can be represented as

$$w(\xi, \eta) = W(\eta)\chi(\xi), \tag{4}$$

where $\chi(\xi)$ is the given function in the $\xi$ coordinate direction. For the first approximation this function can be specified as the deflection of a beam applied by the load $q_\xi$ and associated boundary conditions in the $\xi$ — direction. In order to find the unknown function $W(\eta)$ the Kantorovich variational procedure can be applied to the Eq. (3) as follows

$$\frac{D}{l_y^4}\int_0^1 \left(\gamma^4 \frac{\partial^4 w}{\partial \xi^4} + 2\gamma^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4}\right)\chi(\xi)\,\mathrm{d}\xi = \int_0^1 q(\xi,\eta)\chi(\xi)\,\mathrm{d}\xi, \tag{5}$$

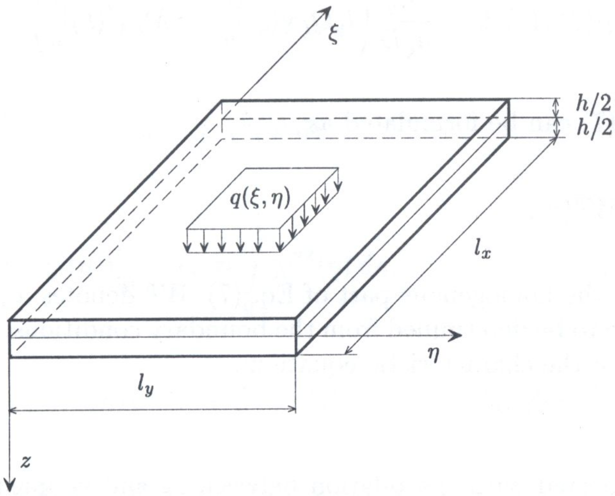



**Fig. 1.** Rectangular thin plate in cartesian coordinate system

where $\gamma = l_y/l_x$. Integration of the Eq. (5) yields

$$\frac{D}{l_y^4}\left(\gamma^4\int_0^1 \frac{\partial^2 w}{\partial \xi^2}\chi''(\xi)\,\mathrm{d}\xi - 2\gamma^2\int_0^1 \frac{\partial^3 w}{\partial\xi\partial\eta^2}\chi'(\xi)\,\mathrm{d}\xi + \int_0^1 \frac{\partial^4 w}{\partial\eta^4}\,\mathrm{d}\xi\right)$$
$$+\frac{D\nu\gamma^2}{l_y^4}\left(\frac{\partial^3 w}{\partial\xi\partial\eta^2}\chi(\xi)\Big|_0^1 + \frac{\partial^2 w}{\partial\eta^2}\chi'(\xi)\Big|_0^1\right)$$
$$+\frac{\gamma}{l_y^2}\left(-l_y\underline{Q_\xi\chi(\xi)}\Big|_0^1 + \gamma\underline{M_\xi\chi'(\xi)}\Big|_0^1\right) = q_\eta(\eta)\int_0^1 q_\xi(\xi)\chi(\xi)\,\mathrm{d}\xi, \tag{6}$$

with

$$M_\xi = -\frac{D}{l_y^2}\left(\gamma^2\frac{\partial^2 w}{\partial\xi^2} + \nu\frac{\partial^2 w}{\partial\eta^2}\right),$$

$$Q_\xi = -\frac{D}{l_y^3}\left(\gamma^3\frac{\partial^3 w}{\partial\xi^3} + (2-\nu)\gamma\frac{\partial^3 w}{\partial\xi\eta^2}\right)$$

as the bending moment and the transverse shear force. The underlined terms in Eq. (6) represent the work of the bending moments and shear forces applied on the edges $\xi = 0$ and $\xi = 1$. Setting the solution (4) into the Eq. (6) the following ordinary differential equation can be obtained

$$W^{IV}(\eta) - 2r_\xi^2 W''(\eta) + s_\xi^4 W(\eta) = G_\xi(\eta) \tag{7}$$

with

$$r_\xi^2 = \frac{b_\xi}{a_\xi}\gamma^2,\quad s_\xi^4 = \frac{c_\xi}{a_\xi}\gamma^4,\qquad a_\xi = \int_0^1 \chi^2(\xi)\,\mathrm{d}\xi, \tag{8}$$

$$b_\xi = \int_0^1 \chi'^2(\xi)\,\mathrm{d}\xi - \nu\chi(\xi)\chi'(\xi)\Big|_0^1,\quad c_\xi = \int_0^1 \chi''^2(\xi)\,\mathrm{d}\xi, \tag{9}$$

$$G_\xi(\eta) = \frac{l_y^4}{Da_\xi} q_\eta(\eta) \int\limits_0^1 q_\xi(\xi)\chi(\xi)\,\mathrm{d}\xi + \frac{\gamma l_y^2}{a_\xi D}\left( l_y Q_\xi \chi(\xi)\Big|_0^1 - \gamma M_\xi \chi'(\xi)\Big|_0^1 \right). \tag{10}$$

The solution of the Eq. (7) can be formulated as

$$W(\eta) = \sum_{i=1}^{4} c_i \zeta_i(\eta) + W^p(\eta), \tag{11}$$

where $\zeta_i$ are solutions of the homogenous part of Eq. (7), $W^p$ denotes a particular solution and $c_i$ are constants, which have to be determined from the boundary conditions. The form of the solutions $\zeta_i$ depends on the roots of the characteristic equation

$$k^4 - 2r_\xi^2 k^2 + s_\xi^4 = 0. \tag{12}$$

Four different cases connected with the relation between $s_\xi$ and $r_\xi$ must be considered. Here we introduce the following two types of solutions. In the case $s_\xi > r_\xi$ the functions $\zeta_i$ can be written as

$$\zeta_1(\eta) = \cosh\alpha_\xi\eta \sin\beta_\xi\eta, \qquad \zeta_2(\eta) = \cosh\alpha_\xi\eta\cos\beta_\xi\eta, \tag{13}$$

$$\zeta_3(\eta) = \sinh\alpha_\xi\eta\cos\beta_\xi\eta, \qquad \zeta_4(\eta) = \sinh\alpha_\xi\eta\sin\beta_\xi\eta, \tag{14}$$

$$\alpha_\xi = \sqrt{\frac{s_\xi^2 + r_\xi^2}{2}}, \qquad \beta_\xi = \sqrt{\frac{s_\xi^2 - r_\xi^2}{2}}. \tag{15}$$

For the second case $s_\xi < r_\xi$ we can obtain

$$\zeta_1(\eta) = \sinh\beta_\xi\eta, \quad \zeta_2(\eta) = \cosh\alpha_\xi\eta, \quad \zeta_3(\eta) = \sinh\alpha_\xi\eta, \quad \zeta_4(\eta) = \cosh\beta_\xi\eta, \tag{16}$$

$$\alpha_\xi = \sqrt{r_\xi^2 + \sqrt{r_\xi^4 - s_\xi^4}}, \qquad \beta_\xi = \sqrt{r_\xi^2 - \sqrt{r_\xi^4 - s_\xi^4}}. \tag{17}$$

As shown in [17] the solution (11) can be determined by use of these two types of functions in most application cases.

For the generalized deflection $W(\eta)$ and rotational angle $\bar{\varphi}_\eta = W'(\eta)$ the corresponding statical plate quantities (bending moment and out-of-plane shear force) can be formulated

$$\bar{M}_\eta = W''(\eta) - \nu r_\xi^2 W(\eta), \qquad \bar{Q}_\eta = W'''(\eta) - (2-\nu) r_\xi^2 W(\eta). \tag{18}$$

In order to find the constants $c_i$ as well as the partial solution of the Eq. (7) the transfer matrix formulation can be used as follows

$$\mathbf{z}(\eta) = \mathbf{K}_E(\eta)\mathbf{z}_0 - \mathbf{f}(\eta, \eta_{F,M}). \tag{19}$$

Here $\mathbf{z}^T(\eta) = [W(\eta), \bar{\varphi}(\eta), \bar{M}_\eta(\eta), \bar{Q}_\eta(\eta)]$ is the vector of the resulting functions, the vector $\mathbf{z}_0^T = [W_0, \bar{\varphi}_0, \bar{M}_0, \bar{Q}_0]$ contains the parameters, which must be obtained from given boundary conditions by $\eta = 0$, $\eta = 1$, $\mathbf{K}_E$ is the transfer matrix and $\mathbf{f}(\eta, \eta_{F,M})$ denotes the influence of given external forces and moments. The transfer matrices for the two cases of solutions considered are given in the Tables 1 and 2. With the values $\alpha_\xi, \beta_\xi$ calculated from Eqs. (15) or (17) the function $W(\eta)$ can be determined by use of Eq. (19). The resulting approximation for the plate bending function (4) provides a good accuracy only in the case $l_y \gg l_x$, cf. [1]. Here we continue the calculations by use the function $W(\eta)$ determined as a given function, $\chi(\eta) = W(\eta)$. The corresponding solution $W(\xi)$ can be obtained from the following ordinary differential equation

$$W^{IV}(\xi) - r_\eta{}^2 W''(\xi) + s_\eta{}^4 W(\xi) = G_\eta(\xi) \tag{20}$$

with

$$r_\eta{}^2 = \frac{b_\eta}{a_\eta}\frac{1}{\gamma^2}, \quad s_\eta{}^4 = \frac{c_\eta}{a_\eta}\frac{1}{\gamma^4}, \qquad a_\eta = \int_0^1 \chi^2(\eta)\,\mathrm{d}\xi, \tag{21}$$

$$b_\eta = \int_0^1 \chi'^2(\eta)\,\mathrm{d}\eta - \nu(\chi(\eta)\chi'(\eta))\Big|_0^1, \quad c_\eta = \int_0^1 \chi''^2(\eta)\,\mathrm{d}\eta, \tag{22}$$

$$G_\eta(\xi) = \frac{l_y^4}{a_\eta D\gamma^4} q_\xi(\xi) \int_0^1 q_\eta(\eta)\chi(\eta)\,\mathrm{d}\eta + \frac{l_y^2}{a_\eta D\gamma^4}\left( l_y Q_\eta \chi(\eta)\Big|_0^1 - M_\eta \chi'(\eta)\Big|_0^1 \right). \tag{23}$$

The solution can be formulated in the same way as that for the $\eta$ coordinate direction, whereas the transfer matrix can be taken from Table 1 or Table 2 by replacing $\xi$ into $\eta$. Thus we can approximate the plate bending function by analytical solutions of the ordinary differential equations with coefficients $r_\xi$, $s_\xi$ or $r_\eta$, $s_\eta$ obtained through the variational procedure. This Vlasov's variational steps can be repeated until $|r_\xi^{i+1} - r_\xi^i| < \epsilon$ and $|s_\xi^{i+1} - s_\xi^i| < \epsilon$ with $i$ as the number of iteration and $\epsilon$ as the desired accuracy. The starting function for the iterations can be obtained from the differential equation (20) setting $r_\eta = s_\eta = 0$

$$W^{IV}(\xi) = G_\eta(\xi). \tag{24}$$

The solution represents the transverse bending line of a beam under generalized load $G_\eta(\xi)$ and boundary conditions.

## 3. Examples

### 3.1. Rectangular plates

In order to verify the computational procedure we consider four different examples, Fig. 2, for which analytical solutions are available. For all examples the beam bending function satisfying the kinematical boundary conditions and load types in the $\xi$ coordinate direction (unit force in the case I, Fig. 2 and unit uniformly distributed load in the cases II–IV) was given as a starting approximation. The corresponding coefficients $\alpha_\xi^0$ and $\beta_\xi^0$ have been obtained from Eqs. (8), (9) through the numerical integration and from Eq. (15), whereas the solutions are specified by $s_\xi > r_\xi$ in all examples considered. For the solution $W(\eta)$ the transfer matrix formulation has been used, which yields in the case of clamped edges and the unit force applied at the point $\eta_F$ (Example I) the following expression

$$W_\eta(\eta;\eta_F) = \begin{cases} K_{wM}(\eta)M_0(\eta_F) + K_{wQ}(\eta)Q_0(\eta_F) - K_{wQ}(\eta - \eta_F) & \eta > \eta_F, \\ K_{wM}(\eta)M_0(\eta_F) + K_{wQ}(\eta)Q_0(\eta_F) & \eta \le \eta_F, \end{cases} \tag{25}$$

$$M_0(\eta_F) = \frac{K_{wQ}(1-\eta_F)K_{\varphi Q}(1) - K_{\varphi Q}(1-\eta_F)K_{wQ}(1)}{K_{wM}(1)K_{\varphi Q}(1) - K_{\varphi M}(1)K_{wQ}(1)}, \tag{26}$$

$$Q_0(\eta_F) = \frac{K_{\varphi Q}(1-\eta_F)K_{wM}(1) - K_{wQ}(1-\eta_F)K_{\varphi M}(1)}{K_{wM}(1)K_{\varphi Q}(1) - K_{\varphi M}(1)K_{wQ}(1)}. \tag{27}$$

**Table 1.** Transfer matrix for the case $s_\xi > r_\xi$

| $s_\xi > r_\xi$ | $\zeta_1 = \cosh \alpha_\xi \eta \sin \beta_\xi \eta$ | $\zeta_2 = \cosh \alpha_\xi \eta \cos \beta_\xi \eta$ | $\zeta_3 = \sinh \alpha_\xi \eta \cos \beta_\xi \eta$ | $\zeta_4 = \sinh \alpha_\xi \eta \sin \beta_\xi \eta$ |
|---|---|---|---|---|
| | $W_0$ | $\bar{\varphi}_0$ | $\bar{M}_0$ | $\bar{Q}_0$ |
| $W(\eta)$ | $K_{ww} = 2\alpha_\xi\beta_\xi\zeta_2 - (1-\nu)r_\xi^2\zeta_4$ | $K_{w\varphi} = \frac{1}{2s_\xi^2}\left[(s_\xi^2 - \nu r_\xi^2)\alpha_\xi\zeta_1 + (s_\xi^2 + \nu r_\xi^2)\beta_\xi\zeta_3\right]$ | $K_{wM} = -\frac{\zeta_4}{2\alpha_\xi\beta_\xi}$ | $K_{wQ} = -\frac{1}{2s_\xi^2}\left(\alpha_\xi\zeta_1 - \beta_\xi\zeta_3\right)$ |
| $\bar{\varphi}(\eta)$ | $K_{\varphi w} = \frac{1}{2}\left[(s_\xi^2 - \nu r_\xi^2)\alpha_\xi\zeta_1 - (s_\xi^2 + \nu r_\xi^2)\beta_\xi\zeta_3\right]$ | $K_{\varphi\varphi} = 2\alpha_\xi\beta_\xi\zeta_2 + (1-\nu)r_\xi^2\zeta_4$ | $K_{\varphi M} = -\frac{1}{2}(\alpha_\xi\zeta_1 + \beta_\xi\zeta_3)$ | $K_{\varphi Q} = K_{wM}$ |
| $\bar{M}(\eta)$ | $K_{Mw} = \frac{1}{2}\left[s_\xi^4 - \nu(2-\nu)r_\xi^4\right]\zeta_4$ | $K_{M\varphi} = \frac{1}{2}\left\{\left[s_\xi^4 - 2(1-\nu)r_\xi^2 s_\xi^2 - \nu^2 r_\xi^4\right]\alpha_\xi\zeta_1 - \left[s_\xi^4 + 2(1-\nu) \times r_\xi^2 s_\xi^2 - \nu^2 r_\xi^4\right]\beta_\xi\zeta_3\right\}$ | $K_{MM} = K_{\varphi\varphi}$ | $K_{MQ} = K_{w\varphi}$ |
| $\bar{Q}(\eta)$ | $K_{Qw} = \frac{1}{2}\left\{\left[s_\xi^4 - 2(1-\nu)s_\xi^2 r_\xi^2 - \nu^2 r_\xi^4\right]\alpha_\xi\zeta_1 + \left[s_\xi^4 + 2(1-\nu) \times s_\xi^2 r_\xi^2 - \nu^2 r_\xi^4\right]\beta_\xi\zeta_3\right\}$ | $K_{Q\varphi} = K_{Mw}$ | $K_{QM} = K_{\varphi w}$ | $K_{QQ} = K_{ww}$ |

**Table 2.** Transfer matrix for the case $s_\xi < r_\xi$

| $s_\xi < r$ | $\zeta_1 = \sinh \beta_\xi \eta$ | $\zeta_2 = \cosh \alpha_\xi \eta$ | $\zeta_3 = \sinh \alpha_\xi \eta$ | $\zeta_4 = \cosh \beta_\xi \eta$ |
|---|---|---|---|---|
| | $W_0$ | $\bar{\varphi}_0$ | $\bar{M}_0$ | $\bar{Q}_0$ |
| $W(\eta)$ | $K_{ww} = (\alpha_\xi^2 - \nu r_\xi^2)\zeta_4 - (\beta_\xi^2 - \nu r_\xi^2)\zeta_2$ | $K_{w\varphi} = \alpha_\xi\left[\alpha_\xi^2 - (2-\nu)r_\xi^2\right]\zeta_1 - \beta_\xi\left[\beta_\xi^2 - (2-\nu)r_\xi^2\right]\zeta_3$ | $K_{wM} = \zeta_4 - \zeta_2$ | $K_{wQ} = \dfrac{\alpha_\xi\zeta_1 - \beta_\xi\zeta_3}{\alpha_\xi\beta_\xi}$ |
| $\bar{\varphi}(\eta)$ | $K_{\varphi w} = \beta_\xi(\alpha_\xi^2 - \nu r_\xi^2)\zeta_1 - \alpha_\xi(\beta_\xi^2 - \nu r_\xi^2)\zeta_3$ | $K_{\varphi\varphi} = \left[\alpha_\xi^2 - (2-\nu)r_\xi^2\right]\zeta_4 - \left[\beta_\xi^2 - (2-\nu)r_\xi^2\right]\zeta_2$ | $K_{\varphi M} = \beta_\xi\zeta_1 - \alpha_\xi\zeta_3$ | $K_{\varphi Q} = K_{wM}$ |
| $\bar{M}(\eta)$ | $K_{Mw} = (\alpha_\xi^2 - \nu r_\xi^2)(\beta_\xi^2 - \nu r_\xi^2)(\zeta_2 - \zeta_4)$ | $K_{M\varphi} = \dfrac{(\alpha_\xi^2 - \nu r_\xi^2)\left[\beta_\xi^2 - (2-\nu)r_\xi^2\right]\zeta_3}{\alpha_\xi} - \dfrac{(\beta_\xi^2 - \nu r_\xi^2)\left[\alpha_\xi^2 - (2-\nu)r_\xi^2\right]\zeta_1}{\beta_\xi}$ | $K_{MM} = K_{\varphi\varphi}$ | $K_{MQ} = K_{w\varphi}$ |
| $\bar{Q}(\eta)$ | $K_{Qw} = (\beta_\xi^2 - \nu r_\xi^2)\left[\alpha_\xi^3 - (2-\nu)r_\xi^2\alpha_\xi\right]\zeta_3 - (\alpha_\xi^2 - \nu r_\xi^2)\left[\beta_\xi^3 - (2-\nu)r_\xi^2\beta_\xi\right]\zeta_1$ | $K_{Q\varphi} = K_{Mw}$ | $K_{QM} = K_{\varphi w}$ | $K_{QQ} = K_{ww}$ |

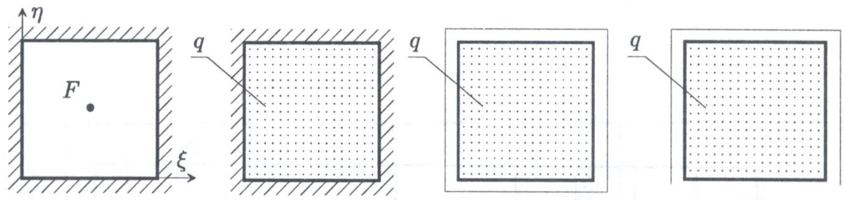



**Fig. 2.** Examples for rectangular plates. I, II — all sides are clamped; III — all sides are simply supported; IV — three sides are simply supported, one side is free

**Table 3.** Convergence of the iterational scheme

| Example | $\gamma$ | Number of iterations | $\alpha_\xi$ | $\beta_\xi$ | $\alpha_\eta$ | $\beta_\eta$ |
|---|---|---|---|---|---|---|
| I | 1 | 4 | 4.325 | 2.274 | 4.325 | 2.274 |
| II | 1 | 3 | 4.156 | 2.416 | 4.156 | 2.416 |
| II | 2 | 4 | 2.293 | 1.603 | 8.295 | 4.616 |
| III | 1 | 3 | 3.178 | 0.174 | 3.178 | 0.174 |
| IV | 1 | 5 | 3.142 | 7.354 | 1.421 | 3.005 |

Such expressions have been formulated for all examples considered for both of the coordinate directions through the specification of two components of the vector $\mathbf{z}_0$ in Eq. (19). The number of Vlasov's variational steps, which is necessary to reach the given accuracy $\epsilon = 1 \cdot 10^{-5}$, is dependent on the type of boundary conditions and loads, Table 3. The least number of iterations has been obtained for the symmetrical problems with the uniformly distributed load (Examples II and III).

For the coeffitients $\alpha_\xi$, $\beta_\xi$, $\alpha_\eta$ and $\beta_\eta$ calculated by use the iterational scheme we can obtain the deflection function $w(\xi, \eta, \alpha_\xi, \beta_\xi, \alpha_\eta, \beta_\eta)$. The results for the deflections and bending moments $M_\xi = -D(\gamma^2 w''_{\xi\xi} + \nu w''_{\eta\eta})/l_y^2$ show a good agreement with analytical solutions, Table 4.

**Table 4.** Comparison with analytical solutions

| Example | $\gamma$ | $w_{max} \cdot 10^3$ | | $M_\xi \cdot 10^2$ | | |
|---|---|---|---|---|---|---|
| | | [16] | our solution | $\xi$, $\eta$ | [16] | our solution |
| I | 1 | $5.60\frac{Fl_y^2}{D}$ | 5.55 | 0, 0.5 | -1.257$F$ | -1.250 |
| II | 1 | $1.26\frac{ql_y^4}{D}$ | 1.263 | 0, 0.5 | -5.13$ql_y^2$ | -5.22 |
| II | 2 | $2.54\frac{ql_y^4}{D}$ | 2.525 | 0, 0.5 | -8.29$ql_y^2$ | -8.45 |
| III | 1 | $4.06\frac{ql_y^4}{D}$ | 4.06 | 0, 0.5 | -4.79$ql_y^2$ | -4.76 |
| IV | 1 | $12.86\frac{ql_y^4}{D}$ | 12.84 | 0.5, 0 | -0.112$ql_y^2$ | -0.110 |

The solution obtained in such a way satisfy exactly the kinematical boundary conditions. The statical boundary conditions are exactly satisfied only in sense of the generalized statical quantities (18) in one of the coordinate directions. For the plates with free edges (e.g. cantilever plates) the statical boundary conditions are satisfied with fairly good accuracy, Fig. 3.

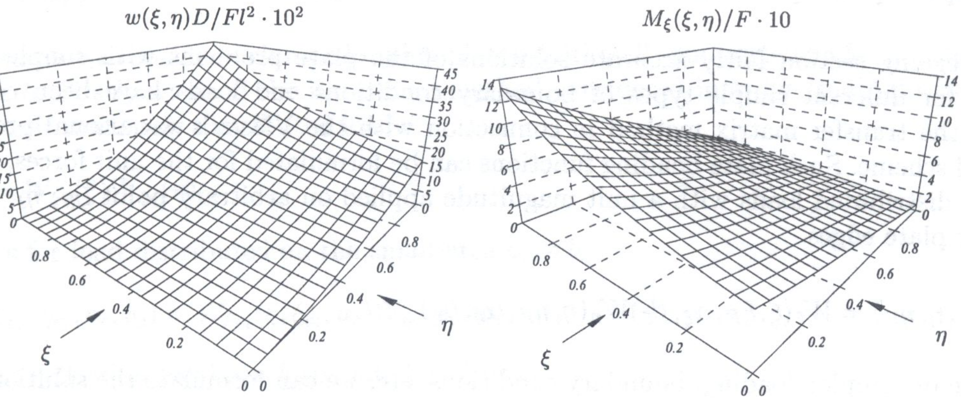
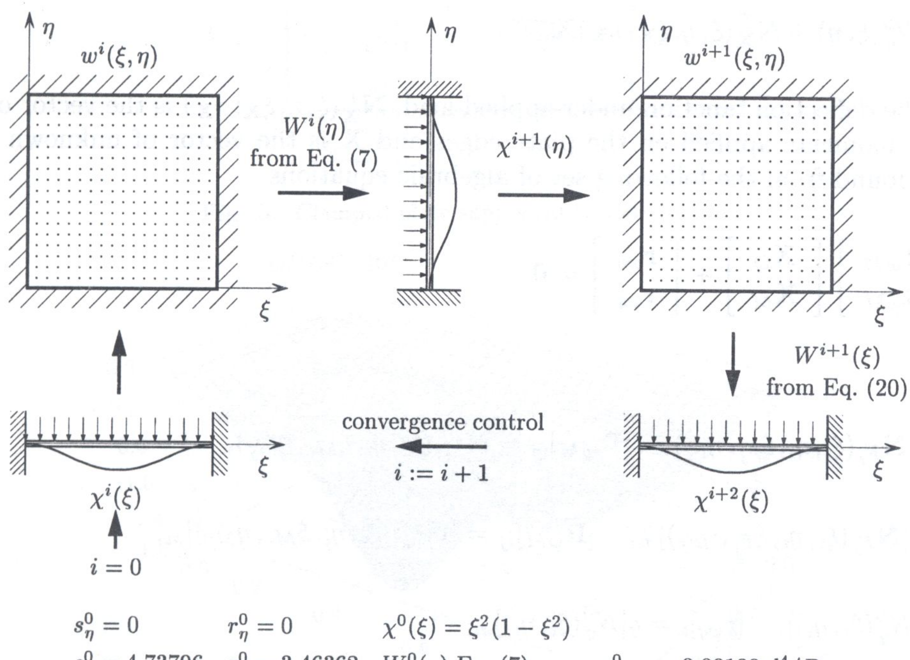



**Fig. 3.** Deflection and bending moments of a cantilever plate (clamped on the edge $\eta = 0$) under force $F$ applied at the point $\xi_F = 1$, $\eta_F = 1$

In addition we compare our results with the solution originally proposed by Vlasov in [17] for the clamped square plate under uniformly distributed load. With the beam function $\chi^0(\xi) = \xi^2(1-\xi)^2$ the values $\alpha_\xi^1 = 4.150$, $\beta_\xi^1 = 2.286$ and the corresponding function $W^1(\eta, \alpha_\xi^1, \beta_\xi^1)$ were obtained, see [17]. Although the resulting maximum deflection $w_{max}^1 = W^1(0.5)\chi^1(0.5) = 0.0013ql_y^4/D$ agree with the analytical solution, the distribution of deflections is non-symmetrical due to the difference of the given and obtained approximations for the $\xi$ and $\eta$ coordinates. The iterational process and the convergence of the solution are illustrated on the Fig. 4. The numerical scheme proposed here improves the solution accuracy and satisfy the symmetry conditions because both of approximations $W(\xi)$ and $W(\eta)$ were obtained as solutions of ordinary differential equations which follow from the variational problem.




| | | | |
|---|---|---|---|
| $s_\eta^0 = 0$ | $r_\eta^0 = 0$ | $\chi^0(\xi) = \xi^2(1-\xi^2)$ | |
| $s_\xi^0 = 4.73796$ | $r_\xi^0 = 3.46362$ | $W^0(\eta)$ Eq. (7) | $w_{max}^0 = 0.00139ql^4/D$ |
| $s_\eta^1 = 4.80736$ | $r_\eta^1 = 3.38692$ | $W^1(\xi)$ Eq. (20) | $w_{max}^1 = 0.00135ql^4/D$ |
| $s_\xi^2 = 4.80737$ | $r_\xi^2 = 3.38695$ | $W^2(\eta)$ Eq. (7) | $w_{max}^2 = 0.00126ql^4/D$ |
| $s_\eta^3 = 4.80737$ | $r_\eta^3 = 3.38695$ | $W^3(\xi)$ Eq. (20) | $w_{max}^3 = 0.00126ql^4/D$ |

**Fig. 4.** Iterational steps and convergence process for the uniformly loaded clamped square plate

### 3.2. Complex plate problems

In the foregoing section fairly accurate solutions of the plate problems with simple rectangular geometry for different simple types of boundary conditions and loads have been composed by means of the transfer matrix method in connection with the Vlasov's variational procedure and iterational scheme. Such plate bending functions can be formulated for the unit forces, moments or uniformly distributed loads with a unit magnitude applied on arbitrary point $\xi_F$, $\eta_F$ of the plate domain or plate edges

$$\bar{w}(\xi,\xi_F,\eta,\eta_F) = W_\xi(\xi,\xi_F,\alpha_\xi,\beta_\xi)W_\eta(\eta,\eta_F,\alpha_\eta,\beta_\eta). \tag{28}$$

In the case of complex loading, boundary conditions, etc. we can formulate the solution as a linear superposition of the deflection functions by unit states characterized by unit forces or moments

$$w(\xi,\eta) = \sum_{i=1}^{n} a_i\bar{w}_i(\xi,\xi_{F_i},\eta,\eta_{F_i}). \tag{29}$$

The unknown coefficients $a_i$ in Eq. (28) can be obtained by minimizing a variational functional formulated for the structure analysed. Such approximations have been used in [5] and [13] for the solutions of creep problems, whereas the governing equations of creeping plates have been transformed into a variational problem with fictious loads and moments calculated from creep strain distribution.

In the case of plate structures (e.g. continuous plates) we can use the method considered in order to formulate the solutions for unit states of simple rectangular domains characterized by unit forces and moments applied on the edges

$$w(\xi,\eta) = N_q^0(\xi,\eta) + \bar{\mathbf{N}}_{\mathbf{X}}^T(\xi,\eta,\xi_{\mathbf{X}},\eta_{\mathbf{X}})\mathbf{X}, \tag{30}$$

where $N_q^0$ is the deflection function under applied load, $\bar{\mathbf{N}}_{\mathbf{X}}^T(\xi,\eta,\xi_{\mathbf{X}},\eta_{\mathbf{X}})$ is the vector of unit states by forces and moments applied on the plate edges and $\mathbf{X}$ is the vector of unknown load factors which can be found from the following set of algebraic equations

$$\begin{bmatrix} \bar{\mathbf{R}}_{wQ} & \bar{\mathbf{R}}_{wM} \\ \bar{\mathbf{R}}_{\varphi Q} & \bar{\mathbf{R}}_{\varphi M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_Q \\ \mathbf{X}_M \end{bmatrix} + \begin{bmatrix} \mathbf{r}_{wq} \\ \mathbf{r}_{\varphi q} \end{bmatrix} = \mathbf{0} \tag{31}$$

with

$$[\bar{\mathbf{R}}_{wQ}]_{ij} = \mathbf{N}_{F_j}(\xi_i,\eta_i,\xi_{F_j},\eta_{F_j}), \quad [\bar{\mathbf{R}}_{wM}]_{ij} = \mathbf{N}_{M_j}(\xi_i,\eta_i,\xi_{M_j},\eta_{M_j}), \tag{32}$$

$$[\bar{\mathbf{R}}_{\varphi Q}]_{ij} = [\mathbf{N}_{F_j}(\xi_i,\eta_i,\xi_{F_j},\eta_{F_j})]_{,n}, \quad [\bar{\mathbf{R}}_{\varphi M}]_{ij} = [\mathbf{N}_{M_j}(\xi_i,\eta_i,\xi_{M_j},\eta_{M_j})]_{,n}, \tag{33}$$

$$[\mathbf{r}_{wq}]_i = q[N_q^0(\xi_i,\eta_i)], \quad [\mathbf{r}_{\varphi q}]_i = q[N_q^0(\xi_i,\eta_i)]_{,n}, \tag{34}$$

$$(\ldots)_{,n} = (\ldots)_{,\xi}n_\xi + (\ldots)_{,\eta}n_\eta \quad i,j = 1,\ldots,m, \tag{35}$$

where $n_\xi$, $n_\eta$ are the components of the unit normal vector to the plate edge and $m$ is the number of the nodes on the boundaries. The following two examples illustrate the application of the Vlasov's approximation in the case of complex boundary conditions and plate geometry.

### *3.2.1. Plate supported on columns*

As a first example we consider a clamped square plate additionally supported on four absolutely stiff columns, Fig 5. Firstly we used the iterational method considered in order to obtain the bending function by unit uniformly distributed load

$$w_q(\xi,\eta) = W_{\xi q}(\xi)W_{\eta q}(\eta) \tag{36}$$

and solutions for load distributed on the small area $a_c \times b_c$

$$\bar{w}_{c_i}(\xi,\eta,\xi_{c_i},\eta_{c_i},a_c,b_c) = W_{\xi c_i}(\xi,\xi_{c_i},a_c)W_{\eta c_i}(\eta,\eta_{c_i},c_c). \tag{37}$$

The resulting deflection has been formulated as follows

$$w(\xi,\eta) = \frac{ql^4}{D}\left[w_q(\xi,\eta) + \sum_{i=1}^{4} X_i\bar{w}_{c_i}(\xi,\eta,\xi_{c_i},\eta_{c_i},a_c,b_c)\right], \tag{38}$$

where the unknown column reactions $X_i$ were calculated by means of the force method. The deflection function, obtained for the case of symmetrically placed columns and $\xi_{c_1} = \eta_{c_1} = 0.3$, $a_c = b_c = 0.05$ is plotted on Fig. 6.

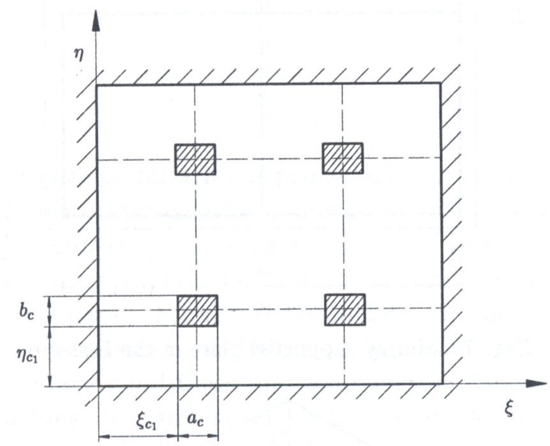



**Fig. 5.** Clamped plate supported on four columns

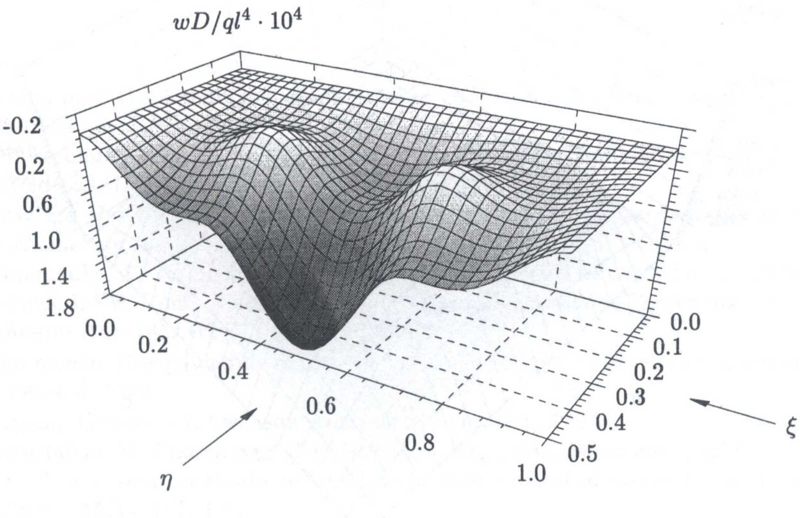



**Fig. 6.** Deflection of a clamped plate supported on columns

### *3.2.2. L-shape plate*

The second example deals with a simply supported plate of the L-shape, Fig. 7. In order to obtain the solution the plate domain has been decomposed into three simple rectangular subdomains with nodal points $i = 1, \ldots m$ on each contact edge. The solution for each of the subdomains is expressed in the form (30), whereas the functions $N_q^0(\xi, \eta)$ and $\bar{\mathbf{N}}_{\mathbf{X}}(\xi, \eta, \xi_{\mathbf{X}}, \eta_{\mathbf{X}})$ have been composed by use the iterational method proposed. Such functions have been plotted in [4]. The unknown forces and moments have been calculated from algebraic set of equations which characterize the compatibility conditions for the deflections and rotation angles in the nodal points. The deflection function obtained by $m = 5$ is plotted on the Fig. 8. This solution shows a good agreement with the ANSYS solution obtained by use of 300 4–node shell elements [6] and the finite difference solution obtained in [15], see Fig. 9.

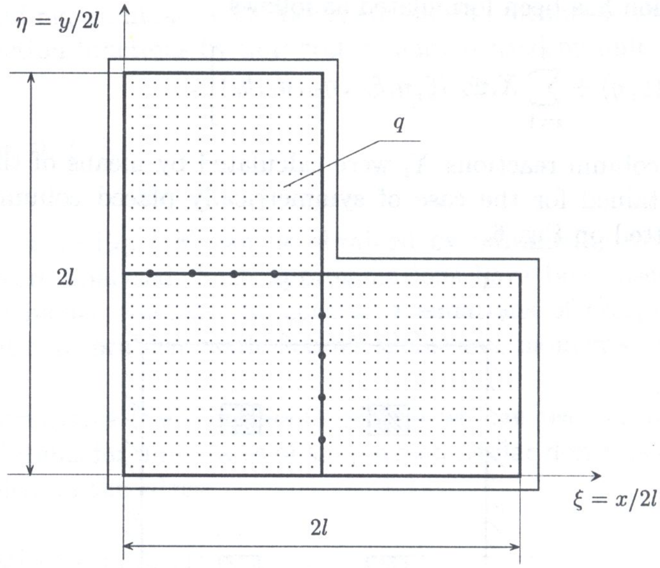



**Fig. 7.** Simply supported plate of the L-shape

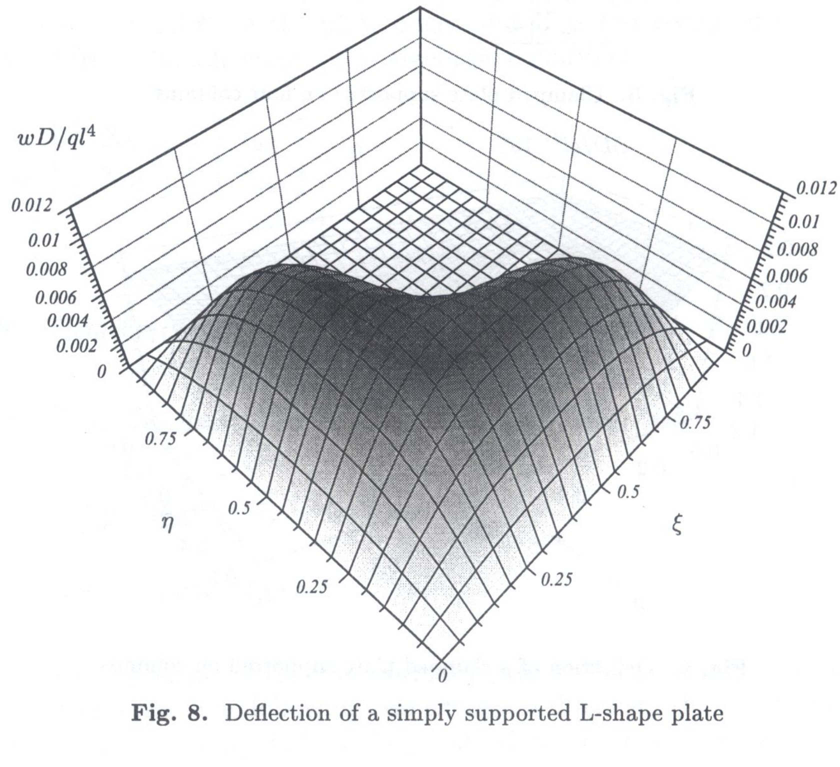



**Fig. 8.** Deflection of a simply supported L-shape plate

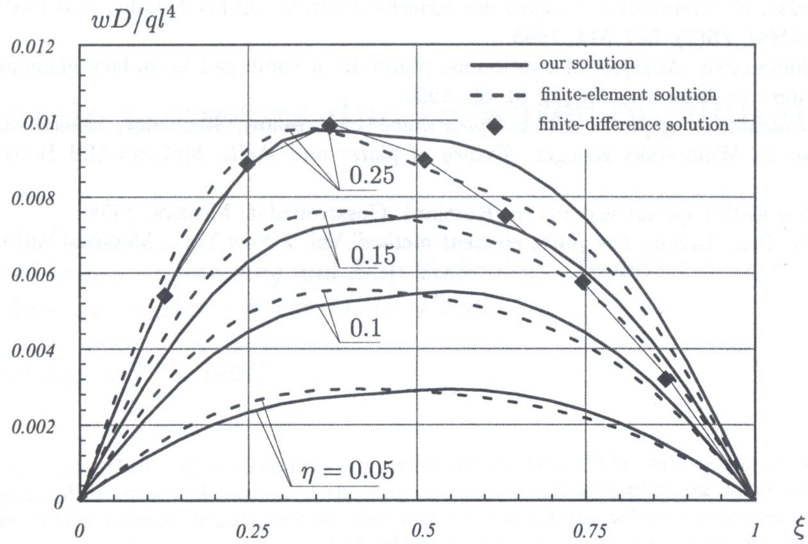



**Fig. 9.** Comparison of solutions for the deflection of a simply supported L-shape plate

## 4. CONCLUSIONS

In the paper presented the Vlasov's variational procedure has been used in connection with the transfer matrix method and an iterational scheme for the solution of the plate bending problems. The deflection function is approximately found as a product of two functions depending on a single coordinate. Such approximation provides a fairly good accuracy in the case of simple geometry, boundary conditions and load factors. Finally the solutions for the complex plate problems have been expressed as a linear superposition of deflection functions composed for rectangular plate domains with simplified boundary conditions and load factors. Such solutions lead to the much small sets of algebraic equations in comparison with domain decomposition methods and need much less computational efforts particulary for the nonlinear problems [5].

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