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Vibrations of a system of two protractile elements with plays taken into consideration

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A description of the play, that may be applied to many other problems, is the basis of the model of a beam system presented in this paper. Equations describing the motion of the system of spring elements under dynamic load have been derived taking into consideration the play occurring between the elements. The equations have been derived according to the Hamilton's variational principle. The play has been treated as a force interacting between the elements whose value depends non-linearly on the mutual distance of the contact places. The function that defines the elements' interacting force may be easily enriched with elements responsible for the energy dissipation, eg. friction. The value of the method presented was proved by the carried out comparative analysis. The equation obtained has been used for an example which the finite elements method (modal technique) has been applied too for comparison. In order to make the comparison more complete, the calculations have been performed not only for the beam model but for a full spatial model as well (basing on the shell model).

1. INTRODUCTION

The behavioural analysis of technical structures is a complex problem due to both the scale and the quantity of occurring phenomena. A correct modelling, i.e. a mathematical mapping, of this complex reality is the basic problem which a contemporary engineer deals with. Despite the existence of many solutions for the general theoretical problems, there are many detailed problems, or specific realizations in technical applications which have to be solved. During the past few years one could notice an especially effervescent development of numerical methods and applications of computer methods which allowed to solve many very difficult problems, or those that could not be solved in other ways. For those problems a correct modelling gets additional importance since, beside the problem of correct reality mapping, there also emerge the problems of numerical presentation, i.e. of discretization. The development of the numerical methods lasting for tens of years has led to some standarizations in the approach to the problems of mechanics. This is clearly evident from the example of the selection of computational methods in which the finite elements method (FEM) and that of boundary elements (BEM) determine a classic approach to most of numerically solved problems, and predominate certainly in practical problems. One should, however, bear in mind that despite the whole comfort of the mentioned numerical methods, many problems may be solved equally effectively, and in many cases even better, either by other methods or approaching a specific problem individually. This paper presents an attempt to model selected processes that occur within a structure of two beams interconnected telescopically. This attempt consists in describing the motion of the beam system, thus it contains a simplified local analysis of the system in that nonlinear phenomena appear in a form taking into consideration the plays at the contact between the system elements. A coupling function in the play zone has been applied that should allow for a free shaping of the play model and the consideration of such phenomena as, for example, friction within the model.

2. THE MODEL OF A TWO-ELEMENT BEAM SYSTEM

Let us try to define an equation which describes the motion of the model of two telescopically interconnected beams (elements) making the following assumptions:

- the strains of the elements are small (the Green's tensor of strains has only linear elements [9];
- the speeds of the strains are low (the material derivative becomes a partial one, and according to the first assumption, the Lagrange's description is in conformity with the Euler's description for low values of the first order [5, 8];
- to concentrate attention, it has been assumed that the load acts only in the plane perpendicular to the beams (axial forces are omitted), and that there is no torsion;
- the interference of the elements results from the forces of reaction during a nonlinear elastic collision.

2.1. Variational formulation of the problem

The Hamilton's variational principle will be used as a starting point for further considerations (this principle has been studied by many authors, e.g. in [6, 7, 10, 13, 15] or [20, 22]), in a form that takes into consideration the occurrence of non-potential forces within the model:

$$\delta H[w_i] \equiv \int_{t_1}^{t_2} \delta(W-K)dt - \int_{t_1}^{t_2}\int_{V} X_i \delta w_i d\nu dt - \int_{t_1}^{t_2}\int_{S_f} x_i \delta w_i ds dt = 0, \tag{1}$$

where:

K – kinetic energy of the system,
W – strain energy of the system,
w_i – components of the field of displacements,
X_i – mass forces,
x_i – boundary forces,
V – area of the body,
S_f – edge of the body on that external loads are imposed.

The Hamilton's variational principle states that, out of all kinematically allowable fields of displacements, only the real, natural field of displacements extremizes the above functional within the time interval from t_1 to t_2. As kinematically allowable is considered a field which is defined for the time moments t_1 and t_2, and within the whole time interval for the area S_u in that the displacements are pre-set. The necessary condition for the extremization of the $\mathbf{H}$ functional is the variation to be equal to zero.

Using the above assumption, it is possible to define the values occurring in formula (1) for the beam.

The elastic energy W is:

$$W = \int_{V} \Phi d\nu = \{V = F \times [0,l]\} = \int_{0}^{l}\int_{F} \frac{1}{2}\sigma\varepsilon df\, dx = \frac{1}{2}\int_{0}^{l} EI\left(\frac{\partial^2 w}{\partial x^2}\right) dx, \tag{2}$$

where:
- Φ – elastic potential,
- σ – field of stresses,
- ε – field of strains,
- EI – beam stiffness,
- $w(x)$ – field of displacements,
- F – cross-sectional area.

The next expression represents the form of the kinetic energy of the beam:

$$K = \frac{1}{2}\int_0^l \left[\overline{m}\left(\frac{\partial w}{\partial t}\right)^2 + I_p\left(\frac{\partial^2 w}{\partial x \partial t}\right)^2\right] dx, \tag{3}$$

where:
- $\overline{m}$ – mass length unit of the beam,
- I_p – mass moment of inertia of the section per unit length.

The inter-element loads acting within the system are presented in the schematic drawing c (Fig. 1).

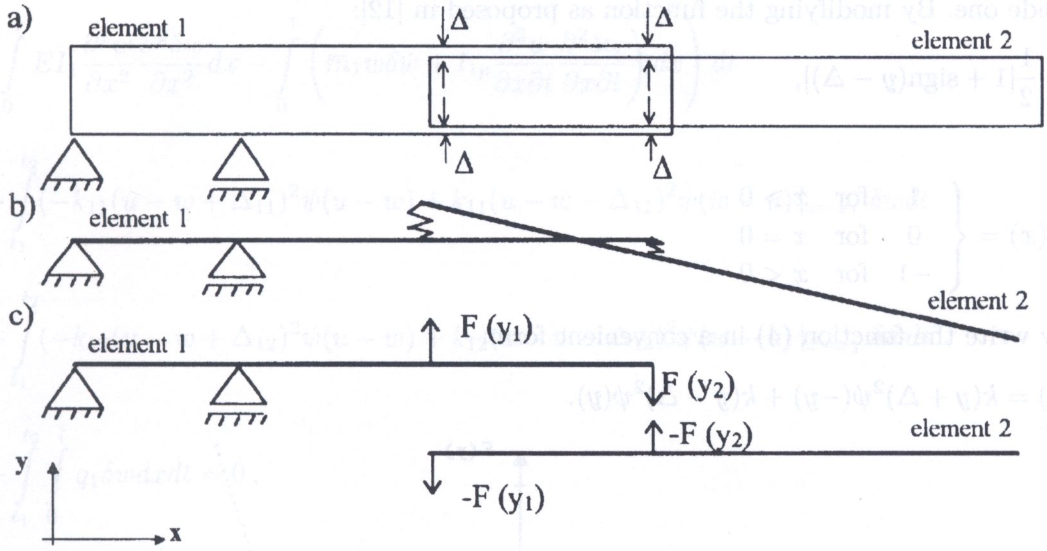

Fig. 1. Schematic drawing of the elements intereference

2.2. Model of the play

As seen in the Figure (Fig. 1), a model has been assumed in which the interference of the elements are replaced by the force of a nonlinear spring. The process of collision is the basic problem in describing the interaction of the two elements. The collision model should provide a piece of information about the values of the collision forces and their duration. Different models are used in the practice (e.g. [11, 19, 20, 27, 30]), for instance the Herz's model, or the more general Sztajerman's model. An analysis of a motion of a rigid body between two bumpers may be found, for example, in [10] or [12]. In this paper, a certain modification of the function suggested by Jones and Muszyńska [12] for a two-mass system model has been used; in the latter model one of the masses moves between two massless bumpers (the occurrence of plays between the mass and the bumpers has been assumed) whereas the other mass performs a flat motion.

Let the function F define the interference force of the spring connecting the elements at the spot of their contact:

$$F(y) = \begin{cases} -k_1\,|y+\Delta|^n & \text{for} \quad y \le -\Delta \\ 0 & \text{for} \quad -\Delta < y < \Delta \\ k_2\,|y-\Delta|^n & \text{for} \quad y \ge \Delta \end{cases}, \tag{4}$$

where:

- y – distance of the co-operating elements,
- k_1, k_2 – constants depending on the elastic properties of the materials, we assume $k_1 = k_2$,
- n – index depending on the assumed collision model, e.g. for the Herz's model $n = 3/2$, in the paper [19] $n = 3$; here, to concentrate our attention and to check the influence of the contact model on the results obtained for the example solved at the end of this article it was assumed that $n = 2$,
- 2Δ – play between the co-operating elements.

One may easily see that, for the index assumed, the function is a continuous one and has a continuous first derivative. Besides, it allows to differentiate the collision model according to the displacement direction of colliding objects, i.e. it allows to distinguish a left-side collision from a right -side one. By modifying the function as proposed in [12]:

$$\Psi = \frac{1}{2}[1 + \mathrm{sign}(y - \Delta)], \tag{5}$$

where:

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases} \tag{6}$$

we may write the function (4) in a convenient form:

$$F(y) = k(y+\Delta)^2\psi(-y) + k(y-\Delta)^2\psi(y). \tag{7}$$

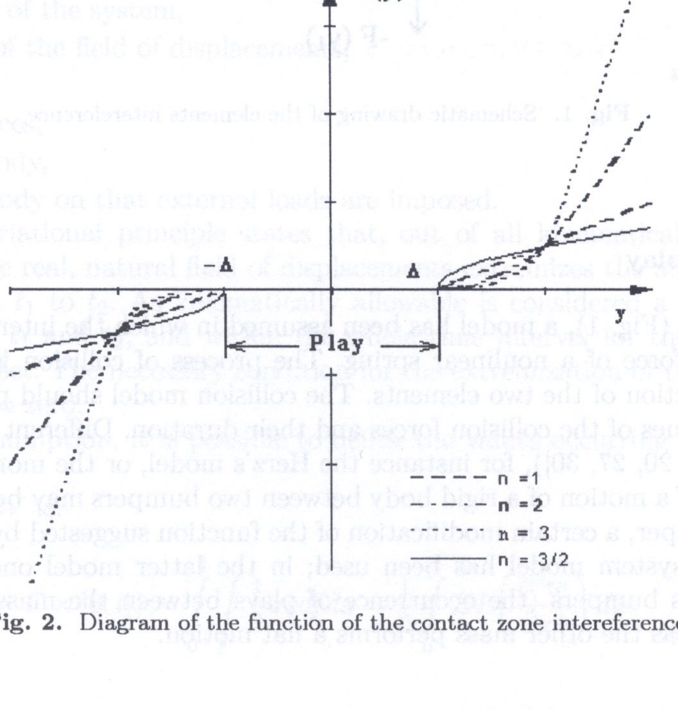

Fig. 2. Diagram of the function of the contact zone intereference

This function makes it possible to take into consideration the friction appearing during the displacement of the elements in a simple form:

$$F(y) = -k(y+\Delta)^2\psi(-y) + k(y-\Delta)^2\psi(y) + \mu N\text{sign}(\dot{x}), \tag{8}$$

where:
μ – coefficient of friction,
N – pressure,
$\dot{x}$ – relative speed of the elements at the spot of contact.

A diagram of the function (4) is presented in Fig. 2. To make the notation more clear, the friction has been omitted in further considerations.

2.3. Variational equations for the two-element system with plays taken into account

For the values assumed it is possible to write the Hamilton's variational principle for the two-element system. The basic derivation of the Hamilton's principle for a beam, using the Rayleigh's equation, may be found in [6, 15, 22] or [30]. Taking into account the Eqs. (1), (2), (3) and (7) notening additionally that also potential forces q are active within the system and making appropriate operations, the following system of two variational equations was obtained:

$$\begin{gathered}
\int\limits_{t_1}^{t_2}\left(\int\limits_0^l EI_1\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2}dx - \int\limits_0^1\left(\overline{m}_1\dot{w}\delta\dot{w} + I_{1p}\frac{\partial^2 w}{\partial x\partial t}\frac{\partial^2 \delta w}{\partial x\partial t}\right)dx\right)dt \\
-\int\limits_{t_1}^{t_2}(-k_{11}(u-w+\Delta_{11})^2\psi(u-w) + k_{11}(u-w-\Delta_{11})^2\psi(w-u)\,|_{x=x_1}\,\delta w dt \\
-\int\limits_{t_1}^{t_2}(-k_{12}(u-w+\Delta_{12})^2\psi(u-w) + k_{12}(u-w-\Delta_{12})^2\psi(w-u)\,|_{x=x_2}\,\delta w dt \\
-\int\limits_{t_1}^{t_2}\int\limits_0^l q_1\delta w dx dt = 0\,, \\
\int\limits_{t_1}^{t_2}\left(\int\limits_{x_0}^{l_1} EI_2\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \delta u}{\partial x^2}dx - \int\limits_{x_0}^{l_1}\left(\overline{m}_2\dot{u}\delta\dot{u} + I_{2p}\frac{\partial^2 u}{\partial x\partial t}\frac{\partial^2 \delta u}{\partial x\partial t}\right)dx\right)dt \\
-\int\limits_{t_1}^{t_2}(k_{21}(u-w+\Delta_{21})^2\psi(u-w) - k_{21}(u-w-\Delta_{21})^2\psi(w-u)\,|_{x=x_1}\,\delta u dt \\
-\int\limits_{t_1}^{t_2}(k_{22}(u-w+\Delta_{22})^2\psi(u-w) - k_{22}(u-w-\Delta_{22})^2\psi(w-u)\,|_{x=x_2}\,\delta u dt \\
-\int\limits_{t_1}^{t_2}\int\limits_{x_0} q_2\delta u dx dt = 0.
\end{gathered} \tag{9}$$

An additonal assumption has been made that both the elements are made of the same material. The following indexing rule is valid in the formula (9): the first index indicates the belonging to the element, the other one shows the place of contact. For values with double indexes the following relation occurs:

$$A_{ij} = A_{(i+1)j} \qquad \text{for} \quad i = 1, \qquad j = 1, 2.$$

Besides, the following denotations were made:

u – allowable kinematic displacement for element 1,

w – allowable kinematic displacement for element 2.

Figure 3 explains the remaining denotations.

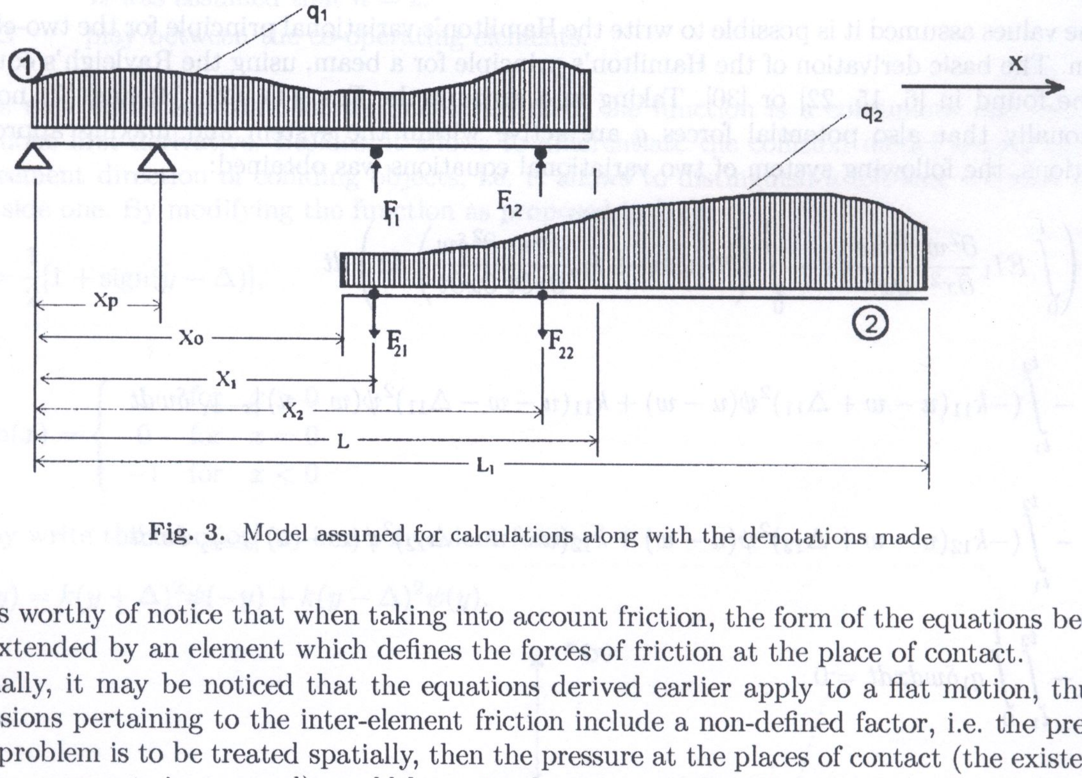

Fig. 3. Model assumed for calculations along with the denotations made

It is worthy of notice that when taking into account friction, the form of the equations becomes only extended by an element which defines the forces of friction at the place of contact.

Finally, it may be noticed that the equations derived earlier apply to a flat motion, thus the expressions pertaining to the inter-element friction include a non-defined factor, i.e. the pressure. If the problem is to be treated spatially, then the pressure at the places of contact (the existence of sliding movements is assumed) would be a simple consequence of the existence of coupling forces as defined by the formula (8). If the previous denotations are assumed, then, adding only indices which define the plane (Fig. 4) to which a given magnitude applies (y for the XY plane, z for the XZ plane), the coupling force within the XY plane can be expressed as:

$$F_y(y) = -k_y(y + \Delta_y)^2\psi(-y) + k_y(y - \Delta_y)^2\psi(y) + \mu_y F_z(z)\text{sign}(\dot{y}) \tag{10}$$

and

$$F_z(z) = -k_z(z + \Delta_z)^2\psi(-z) + k_z(z - \Delta_z)^2\psi(z) + \mu_z F_y(y)\text{sign}(\dot{z}) \tag{11}$$

for the XZ plane where all the denotations are as previously and of the introduced indexation scheme is maintained.

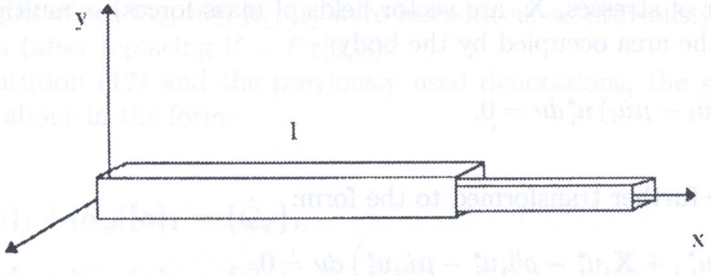

Fig. 4. The applied coordinate system

3. Methods of solving the problem of the dynamics of a two-element system

The best way to solve Eqs. (9) is to use the direct methods. There is a number of methods available. The Ritz's method is one of the most common ones, examples may be found in many papers, e.g. [7, 10, 14, 15, 23, 24, 25]. A version of the generalised Ritz's method, often used in dynamic analyses is called the Kantorowicz's method [7, 10, 15, 23] in which the solution of Eqs. (9) is adopted in the following form:

$$w^{(N)}(x,t) = \sum_{i=1}^{N} a_i(t)\varphi_i(x). \tag{12}$$

A disadvantage of this approach is the necessity to solve a system of differential equations. On the other hand, algebraic equations have to be solved using the Ritz's method.

3.1. The Kantorowicz's method

To derivate the equations, it is most convenient to use the approach applied in [28], e.g. in the finite elements method. This simplifies the problem given by Eqs. (9) to a statical one (Lagrange's functional) in which dynamic interferences are taken into account by enriching the mass forces by forces of inertia and forces of damping. In the index denotation it is:

$$-\rho\ddot{u}_i - \mu\dot{u}_i \tag{13}$$

Here, to make things simpler, the rotational inertia of the section is not taken into consideration. In expression (13), the first element represents the forces of inertia (ρ – density), whilst the second one represents the forces of damping (μ – coefficient of damping). In order to derivate the Lagrange's function, it is possible to define the functional space whose elements will constitute vectors of kinematically allowable displacements u^*, (Berdyczewski [1]). Such a space is defined as follows:

$$W^* = \{u_i^* : u_i^*(x_j,t) = \tilde{u}_i = 0, \quad x_j \in S_f, t \in < t_1, t_2 >; \quad i,j = 1,2,3,...^*\}. \tag{14}$$

Elements belonging to the boundary S_f, on which there are some defined displacements, are denoted with the mark $\sim$ in the definition (14). If any element u_i^* belonging to the space of kinematically allowable displacements W^* is taken, and the equation of equilibrium (using the Einstein's summation convention):

$$\sigma_{ij,j} + X_i = \rho\ddot{u}_i + \mu\dot{u}_i \tag{15}$$

(where σ_{ij} is a tensor of stresses, $\mathbf{X}_i$ are vector fields of mass forces) is multiplied by this element and integrated over the area occupied by the body:

$$\int_V ((\sigma_{ij,j} + \mathbf{X}_i - \rho\ddot{u}_i - \mu\dot{u}_i)\, u_i^* d\nu = 0, \tag{16}$$

this equation may be further transformed to the form:

$$\int_V \Big((\sigma_{ij}u_i^*)_{,j} - \sigma_{ij}u_{i,j}^* + \mathbf{X}_i u_i^* - \rho\ddot{u}_i u_i^* - \mu\dot{u}_i u_i^*\Big)\, d\nu = 0. \tag{17}$$

Further transformations may be carried out applying the Green's theorem to the first term on the left side and considering the boundary conditions ($u_i^* = 0$), while the definition of a linear tensor of strains is applied to the second term of the expression (17). Taking into account the above considerations, the following equation is obtained:

$$\int_V \sigma_{ij}\varepsilon_{ij}^* d\nu = \int_V (\mathbf{X}_i - \rho\ddot{u}_i - \mu\dot{u}_i)u_i^* d\nu + \int_{S_f} x_i u_i^* ds. \tag{18}$$

In this equation the denotations are the same as in the formula (1). Kinematically allowable displacements u^* should be treated as variations of displacements. At this place, it is worthwhile noting that the variational equation obtained may be written in the form:

$$a(u_i, u_i^*) = f(u_i^*) \qquad ,\forall_{u_i^* \in W^*}, \tag{19}$$

in that $a(ui, ui^*)$ is the left-hand side of the Eq. (18), and $f(ui^*)$ is the right-hand side of the same equation. As may be easily checked, the functional obtained is a variation of the functional in the following form:

$$J(w) = \frac{1}{2}a(w,w) - f(w). \tag{20}$$

If, further on, for Eq. (18) an procedure analogical to that of deriving the expression (9) is used, the searched form of the Lagrange's functional variation for the problem of two moving, overlapping beam elements is obtained. The zeroing of the variation is a necessary condition to extremize the functional. For the problem shown by Eq. (18), taking into account the inter-element forces, it is as follows:

$$\begin{aligned}
&\int_0^l EI_1 \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2}dx - (-k_{11}(u-w+\Delta_{11})^2\psi(u-w) + k_{11}(u-w-\Delta_{11})^2\psi(w-u)\,|_{x=x_1}\,\delta w \\
&\quad -(-k_{12}(u-w+\Delta_{12})^2\psi(u-w) + k_{12}(u-w-\Delta_{12})^2\psi(w-u)\,|_{x=x_2}\,\delta w \\
&\quad -\int_0^l q_1\delta w dx + \int_0^l \overline{m}_1\ddot{w}\delta w dx + \int_0^l \overline{\mu}_1\dot{w}\delta w dx = 0, \\
&\int_{x_0}^{l_1} EI_2 \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \delta u}{\partial x^2}dx - (k_{21}(u-w+\Delta_{21})^2\psi(u-w) + k_{21}(u-w-\Delta_{21})^2\psi(w-u)\,|_{x=x_1}\,\delta u \\
&\quad -(k_{22}(u-w+\Delta_{22})^2\psi(u-w) + k_{22}(u-w-\Delta_{22})^2\psi(w-u)\,|_{x=x_2}\,\delta u \\
&\quad -\int_{x_0}^{l_1} q_2\delta u dx + \int_{x_0}^{l_1} \overline{m}_2\ddot{u}\delta u dx + \int_{x_0}^{l_1} \overline{\mu}_2\dot{u}\delta u dx = 0.
\end{aligned} \tag{21}$$

In Eq. (21) the denotations $\overline{m}_1$, $\overline{m}_2$, $\overline{\mu}_1$, $\overline{\mu}_2$ were introduced, as previously, for magnitudes per a unit of beam length (after replacing $V = Fx[0,l]$).

Next, using substitution (12) and the previously used denotations, the system of differential equations is brought about in the form:

$$[M_\varphi]\{\ddot{a}\}_1 + [C_\varphi]\{\dot{a}\}_1 + [K_\varphi]\{a\}_1 = \{\hat{Q}_\varphi\},$$
$$[M_\varphi]\{\ddot{a}\}_2 + [C_\varphi]\{\dot{a}\}_2 + [K_\varphi]\{a\}_2 = \{\hat{Q}_\varphi\}, \tag{22}$$

where:

$$-\text{stiffness matrices:} \quad [K_\varphi] = \int_0^l \{\varphi''\}^T EI_1\{\varphi''\}dx, \quad [K_\varphi] = \int_{x_0}^{l_1} \{\phi''\}^T EI_2\{\phi''\}dx, \tag{23}$$

$$-\text{ matrices of masses:} \quad [M_\varphi] = \int_0^l \{\varphi\}^T \overline{m}_1\{\varphi\}dx, \quad [M_\varphi] = \int_{x_0}^{l_1} \{\phi\}^T \overline{m}_2\{\phi\}dx, \tag{24}$$

$$-\text{matrices of damping:} \quad [C_\varphi] = \int_0^l \{\varphi\}^T \overline{\mu}_1\{\varphi\}dx, \quad [C_\varphi] = \int_{x_0}^{l_1} \{\phi\}^T \overline{\mu}_2\{\phi\}dx, \tag{25}$$

$$-\text{matrices of load:} \quad \{\hat{Q}_\varphi\} = \int_0^l q_1\{\varphi\}^T dx + f_\varphi(\{a\}_1, \{a\}_2), \quad \{\hat{Q}_\phi\} = \int_{x_0}^{l_1} q_2\{\phi\}^T dx \tag{26}$$
$$+ f_\phi(\{a\}_2, \{a\}_1).$$

In the latter ones (26), the following definition of the vector of inter-element interferences has been adopted:

$$f_\varphi = F(\{a\}_2^T\{\phi\} - \{a\}_1^T\{\varphi\}\,|_{x=x_2}\, \{\varphi\}^T\,|_{x=x_2} + F(\{a\}_1^T\{\varphi\} - \{a\}_2^T\{\phi\}\,|_{x=x_2}\, \{\varphi\}^T\,|_{x=x_1}\,,$$
$$f_\phi = F(\{a\}_2^T\{\phi\} - \{a\}_1^T\{\varphi\}\,|_{x=x_1}\, \{\phi\}^T\,|_{x=x_1} + F(\{a\}_1^T\{\varphi\} - \{a\}_2^T\{\phi\}\,|_{x=x_2}\, \{\phi\}^T\,|_{x=x_2}\,. \tag{27}$$

The remaining denotations are as previously but the solution is approximated by Eq. (12). It is worthy of notice here that in a specific case of motion, e.g. that of a telescopic jib, the damping may only be defined at the places at which sliding movements for dry friction occur:

$$F_t = -\mu N \operatorname{sign}(\dot{u} - \dot{w})\,|_{x_i}\,, \tag{28}$$

where:

- F_t – force of dry friction,
- N – force of pressure of the elements on one another,
- μ – coefficient of friction,
- $\dot{u}$, $\dot{w}$ – speeds of the first and second elements,
- x_i – coordinate of the i-th sliding motion.

The system of Eq. (22) may be conveniently solved using, for instance, the finite differences method, or the base method after imposing appropriate boundary conditions (the base functions must be kinematically allowable) [16, 17] and [18].

4. EXAMPLES

To depict the considerations discussed, an attempt has been undertaken to solve an exemplary system of two beam elements, using a couple of methods. The following input data have been adopted (see Fig. 5):

Young's modulus $E_{yz} = 2.110^8$ kN/m^2,
moment of inertia of the first beam section $J_{y1} = 3.798610^{-4}$ m^4,
moment of inertia of the second beam section $J_{y2} = 2.182910^{-4}$ m^4,
unit mass of the first beam $m_{p1} = 83.16$ kg/m,
unit mass of the second beam $m_{p2} = 55.07$ kg/m,

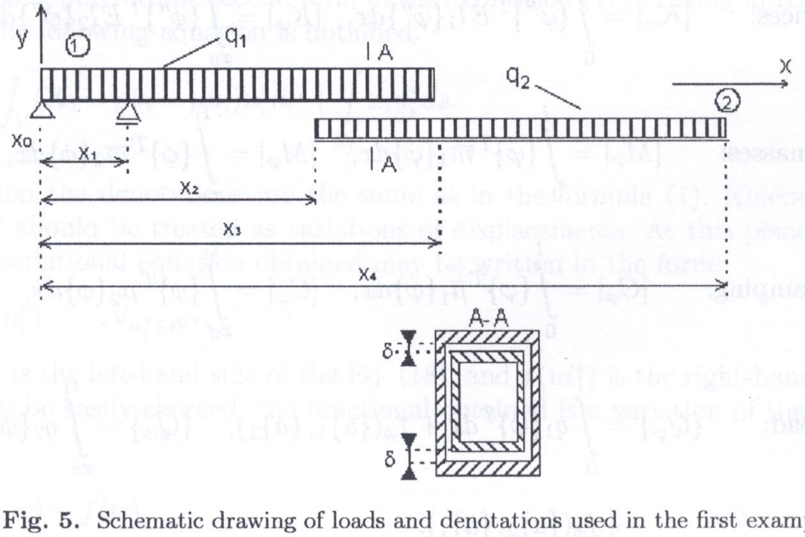

Fig. 5. Schematic drawing of loads and denotations used in the first example

unit mass moment of inertia of the first beam $I_{p1} = 3.4810^{-3}$ kgm,
unit mass moment of inertia of the second beam $I_{p2} = 1.6810^{-3}$ kgm,
spring constant of the interference of colliding elements $k = 1010$ kN/m,
inter-element play $\delta = 0.014$ m,
$x_1 = 2.0$ m,
$x_2 = 9.0$ m,
$x_3 = 10.0$ m,
$x_4 = 19.0$ m.

Loads:
$q_1 = 83.16$ N/m,
$q_2 = 55.07$ N/m.

Initial conditions:
displacements and zero speeds in the coordinate system adopted. For the first calculations a lack of friction has been assumed, then its occurrence has been taken into consideration.

4.1. Vibrations of the system when friction is neglected

In the Kantorowicz's method potential series has been adopted with expressions defined as follows:
a) for the first term

$$\varphi^i(x) = x^i(x - x_0)(x - x_1) \tag{29}$$

b) for the second term (the first two expressions):

$$\phi^i(x) = (x - x_2)^{(i-1)}, \quad i < 3 \tag{30}$$

(the consecutive terms)

$$\phi^i(x) = (x - x_2)^{(i-1)}(x - x_4), \quad i > 2. \tag{31}$$

It is worthy of notice that the first two terms of the series (30) provide a solution if the second element moves as a rigid body. For the first numerical experiment an approximate solution has been adopted in the form of the first two terms of the series, i.e.

– function of the deflection of the first element:

$$w(x,t) = a_1(t)\varphi^1(x) + a_2(t)\varphi^2(x), \tag{32}$$

– function of the deflection of the second element:

$$u(x,t) = a_3(t)\phi^1(x) + a_4(t)\phi^2(x), \tag{33}$$

where the functions ϕ^i, φ^i are defined by the Eq. (29) and (30). The problem of the form (22) has been solved numerically. A mathematical software has been used allowing for a simple application of the equations (MATHEMATICA 2.2). In the applying of the system, however, it was difficult to obtain numerically stable solutions for a period of times longer than a few seconds, especially when friction was taken into consideration. In order to check the convergence of the solution, Eq. (22) has also been solved for base functions reflecting the deflection of the second element. The series of base functions has been based on the potential series (29)–(31) from which the first two expressions, the Eq. (32) and the first three expressions have been selected as an approximate of the solution for the first and second elements, respectively:

$$u(x,t) = a_3(t)\phi^1(x) + a_4(t)\phi^2(x) + a_5(t)\phi^3(x), \tag{34}$$

where the functions ϕ^i, φ^i are defined analogically as previously.

An exact comparison of the two solutions using four and five terms in the series shows that when the deformability of the second element is taken into account a change occurs in the solution starting with the first collision of the elements. This is justified since until that moment, the second element moves as a rigid body due to the lack of constraints which could bring about its deformation. At the end of the time interval taken into consideration, the divergence of the two solutions exceeds 20%.

Two models of the problem under consideration have been constructed comparatively, using the finite elements method; the first model is a classical beam model, the other one is a spatial model basing on shell elements. As a reference for the solutions of the problem obtained by different methods, the results obtained by the finite elements method for the beam system without taking into account the play occurring in combination have been presented. In all the examples solved using the finite elements method, the COSMOS/M system ([2, 3, 4] and [21]) was used.

4.2. Summary of the solutions obtained for vibrations with neglected friction

To illustrate the calculations carried out, it is convenient to compare the time runs of selected points of the system obtained by different methods.

The element ends are doubtless characteristic points; especially the end of the second element is a precious point of observation since even slight differences at the contact spots of the elements should get multiplied at that end. The picture below (Fig. 6) presents the characteristic curve described.

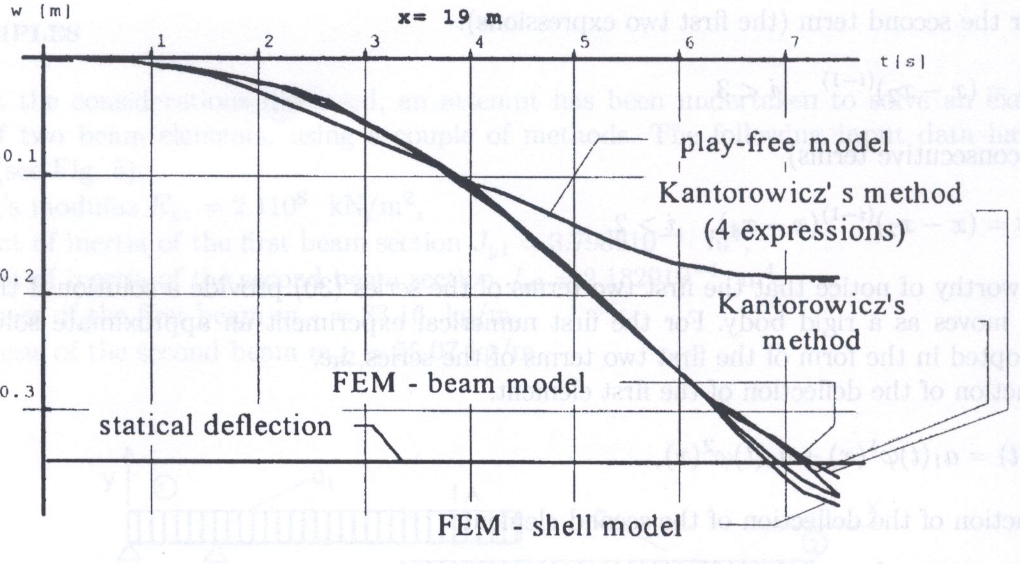

Fig. 6. Comparison of the system end trajectory obtained by different analysis methods

A large conformity of the motion runs during the first six seconds of the motion is evident in the diagram. Next, a clear stratification of the curves obtained by different solution methods follows. The difference remain, however, minor. It may be noticed that the deflection is growing with increasing complication of the model (mapping accuracy). The diagram also shows a fundamental difference between the solutions of the vibration problem obtained with and without taking into account the play between the elements.

The end of the first element is another point of comparison. The behaviour of this point makes it also posible to compare the solutions obtained. The diagram (Fig. 7) presents the behaviour of this point during the motion of the system.

A large conformity of the solution runs is indicated in Fig. 7. Relative differences are greater than in the previous characteristic curve, however, the absolute differences are of the same level. In both models, the neutral axis of the elements has been chosen as a reference for the shell model.

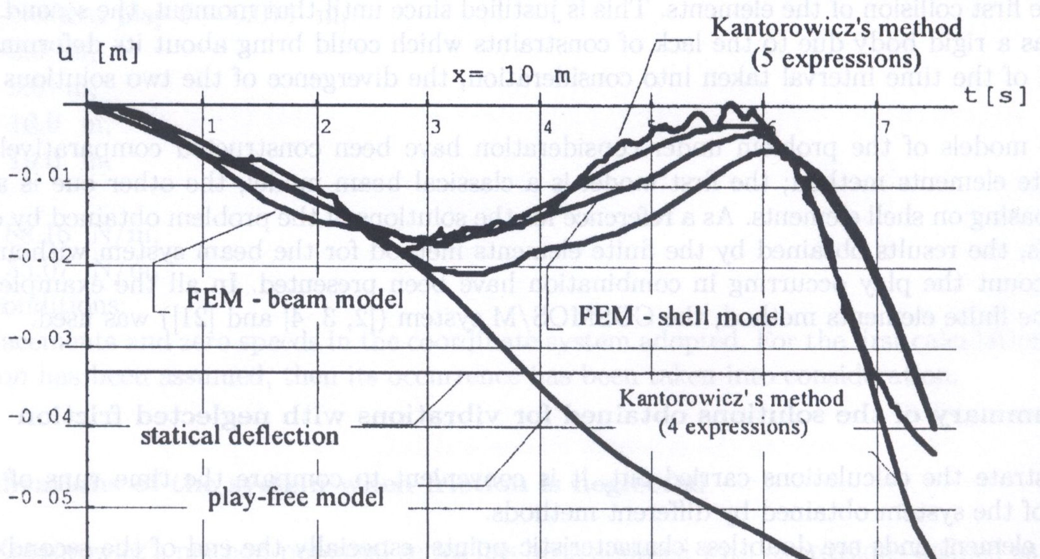

Fig. 7. Comparison of the trajectories of the end of the first element resulting from different methods of analysis

4.3. Solution with friction taken into account

In the Kantorowicz method, a potential series, with expressions defined like those for the example without friction, was adopted as the base functions, (32), (33). In the numerical experiment, an approximate solution in the form of the first two expressions of series (32), (33) has been adopted, where the functions ϕ^i, φ^i are defined by the Eqs. (29), (30) and (31). The matrices of masses and those of stiffness are identical as in the case of the friction-free analysis. In Eq. (22), only the element describing the inter-element interference (27) has been extended by the friction in the zone of sliding movements, Eq. (28). In this example, the same coefficient of friction μ has been assumed for all the sliding movement:

$\mu = 0.2$

Pressures in the contact zone (a flat model was assumed):

a) for $x = x_3$:

$N1 = 5000[\text{N}]$,

b) for $x = x_2$:

$N2 = 4500[\text{N}]$.

To check the convergence and to make the comparsion complete, Eq. (22) has been solved also for base functions reflecting the deflection of the second element. The series of base functions has been based on the potential series from which, as an approximation of the solution, the first two expressions have been selected for the first element, and Eq. (32) and the first three expressions for the second element (see (34)). The functions ϕ^i, φ^i are defined, analogically as previously, by the Eqs. (29), (30) and (31). For the new approximation of the solution, the necessary matrices defined by equations (22) to (27) are identical as in the example without friction. Only Eq. (27) is enriched with the element (28). The problem defined by equations (9) has been solved numerically using, as given previously, a function of the MATHEMATICA 2.2 program.

In the comparative models, the considerations have been limited to the beam model only. The necessity of taking friction into account has brought about the need to change the beam model. The two-dimensional, flat elements have been replaced by elements having a full description, i.e. six degrees of freedom in a node: three translational ones, u_x, u_y, u_z, and three rotational ones α_x, α_y, α_z. To model friction in the model so that it could comply with the assumptions about the flat motion of the model was troublesome since for the friction model the model analysis requires the consideration of displacements in the direction perpendicular to the contact surface of interference of the bodies analysed. This results in an expanding of the analysis by proper frequencies for vibrations perpendicular to the basic analysis. This makes the model not fully consistent with those presented previously. The definition of the contact elements makes it possible to take into account only the friction that occurs within the system. During the numerical analysis of the problem it turned out, however, that the stability of the proposed of way solution has been weak using the modal technique and depends strongly on the occurring forces of friction. It is of interest that, the stability of the solution has not been influenced to a considerable degree by the increasing number of integration steps. The problem of the solution consisted in a strong impact of the force of friction on the construction deformations at a given integration step. Finally, a solution has been obtained for the first 7 seconds of the motion with 4375 integration steps.

4.4. Summary of the calculations with the impact of friction taken into account

A comparison of the time runs of selected system points obtained by different methods illustrates the calculations carried out. As in the previous example, the element ends, especially the second element end, are characteristic points since even minor differences at the contact place of the elements should get multiplied at those ends. Figure 8 presents the described characteristic curve.

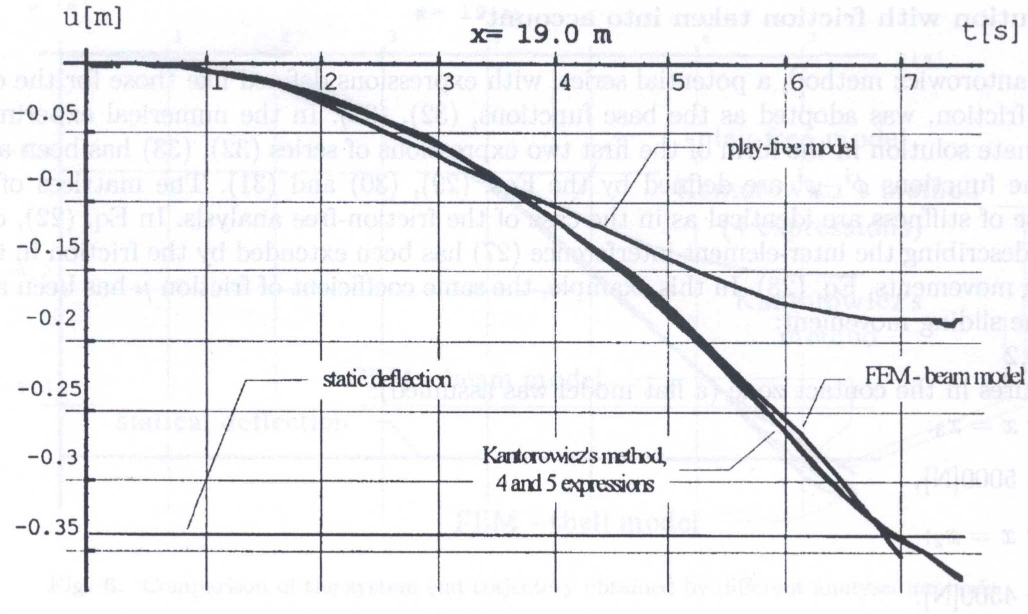

Fig. 8. Comparison of the trajectories of the system end obtained from different analysis methods

A large conformity of the motion run during the first six seconds of the motion is evident in the diagram. Then some stratification of the curves obtained from different solution methods follows. The differences remain, however, minor. The solution obtained taking into account the occurrence of plays in the jib model deviates considerably from the "play-free" model. It is evident that, in order to obtain precise dynamic runs, it is necessary to consider the occurrence of plays. The first element end is another point of comparison. Taking this point into account allows to compare the solutions obtained. The following diagram (Fig. 9) presents the behaviour of the point being discussed during the motion of the system.

The shape conformity of the solution runs is evident in Fig. 9. The relative differences are, however, large and considerably higher than those of the previous characteristic, but with absolute differences are comparable. Only the solution run for the model ignoring plays deviates considerably from the remaining characteristics. Not only the deflection values but also the nature of the run are quite different.

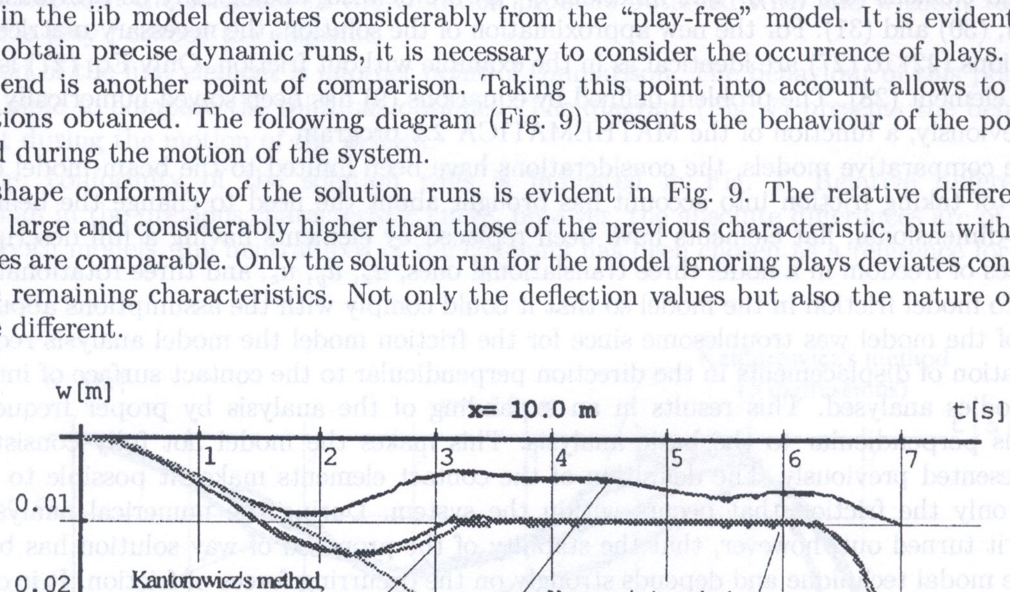

Fig. 9. Comparison of the first element end trajectory with different analysis methods

The differences that may be noticed between the run of the comparative solution using the finite elements method and that of the solution of the model described in this paper result from the need to consider, together with the series of proper frequencies of the model, those frequences that vibrations are perpendicular to the fundamental plane of vibrations.

5. Conclusions

A series of conclusions and remarks worthy of notice result form the considerations that have been carried during modelling the system of beams combined with plays. It may be stated on the basis of the analysis carried out that:

1. The variational equations derived in this paper in order to describe the behaviour of a flexible system of beams interconnected by two connections of a point-type nature, where the occurrence of plays have been taken into account, are a mathematical equivalent of a specific technical application, such as, for example, a telescopic jib. It should be added that the described phenomena appear widely in technical devices and the applied description may be easily adopted.

2. The function of the elements interference, that was introduced when deriving the equations of the behaviour of the system of beams, has permitted a concise mathematical description of the collision of two flexible elements and provides the capability of obtaining full information on the run of this event; it allows also to modify the same model of collision by a change of the parameters of the function of the elements interference.

3. The behaviour of the two-element system described in this work is not a closed solution. It allows to broaden the description onto systems having more than two elements. This is possible and easy due to the combination of the elements with an interference function having the nature of a force constraint. Each next element means two additional variational equations describing its behaviour in two vibration planes combined with the remaining ones by an inter-element interference.

4. The proposed model permits also the consideration of damping at the spots of connection of the elements. This has been proved for an example by introducing the force of friction into the contact zone of the elements.

5. The new equations introduced in this paper describe correctly the complex behaviour of beams in an operation in that the dynamics of the system with plays is taken into account. As proven by the numerical experiments obtained in this way, the results are in conformity with those obtained from comparative models.

6. A comparison of the deflection run in the play-free model and in that taking plays into account allows to state that the neglecting of the occurrence of plays brings about a considerable change in the results obtained, thus it may constitute a source of serious errors.

7. It is worthwhile noting that in the examples of solutions a clear impact of the model of collision on the deflection function has not been observed. In the models that have been created to meet the needs of the finite elements method another model of collision has been used that corresponds with the model described by Eq. (4) with the exponent $n = 1$.

8. The assumption of the continuous function of the elements interference has result in a lower cost of numerical calculations. The results presented for the models suggested in this work were obtained with, at highest, 2000 integration steps whilst the least expensive calculations using the finite elements method required more than 4000 intgeration steps.

9. When looking at the comparison of the results of different solution methods, it should be born in mind that the algorythmic approach presented in the form of the finite elements method provides a much larger capability of complication or individualization of the model. There also exists the possibility to consider a series of phenomena that have been omitted until now. Nevertheless, the solution obtained certainly gives interesting information on the dynamics of the system under consideration treated as a whole. An important point of the Kantorowicz's methods is the possibility to obtain satisfactory results using a simple base function.

References

[1] J. Berdychevskij. *Variational Principles of Continuum Miechanics* [in Russian]. Nauka, Moskva 1983.

[2] Cosmos/M. *Basic FEA System. Linear-Static, Natural Frequency and Buckling Analysis.* ver. 1.70 vol. 3, Santa Monica, California, 1993.

[3] Cosmos/M. *Advanced Modules User Guide. Advanced Dynamics, Heat Transfer and Nonlinear Static & Dynamic Analysis.* ver. 1.70 vol. 4, Santa Monica, California, 1993.

[4] Cosmos/M. *Finite Element Analysis System Advanced Seminar. Structural Research & Analysis Corporation.* Santa Monica, California, 1993.

[5] A.C. Eringen. *Nonlinear Theory of Continuus Media.* Wiley, New York, 1962.

[6] Y.C. Fung. *Foundations of Solid Mechanics* (Polish translation), 3rd Ed., PWN, Warszawa, 1969.

[7] I.M. Gelfand, S.W. Fomin. *Variational Calculus* (Polish translation), 3rd Ed., PWN, Warszawa, 1975.

[8] A. Green. *Foundations of Nonlinear Theory of Continua* (Polish translation), Wrocław, 1973.

[9] A. Green, J.E. Adkins. *Large elastic deformation and nonlinear continuum mechanics.* Oxford University Press. Clarendon Press. London, 1969.

[10] H.M. Hansen, P.F. Chenea. *Mechanics of Vibration.* John Wiley & Sons, Inc., N. Y., 1952.

[11] M.T. Huber. *Theory of Elasticity* [in Polish], PWN, Warszawa 1954.

[12] D.I.G Jones, A. Muszyńska. Harmonic Response of a Damped Two-Degree of Freedom System with Gaps. *Nonlinear Vibration Problems*, PWN, Warszawa, 1979.

[13] O.R. Kajumow. *On the global integrability of certain Lagrangean systems* [in Russian], Izv. AN SSSR: *Mech. of Solids* **6**: p. 16.

[14] M. Kleiber. *Finite element Method in Nonlinear Continuum Mechanics* [in Polish], PWN, Warszawa, 1985.

[15] T. Kozłowski, S. Piechnik, Z. Stojek. *Application of Variational Calculus to the Problems of Structural Mechanics* [in Polish], Arkady, Warszawa, 1967.

[16] J. Kruszewski, E. Wittbrodt, Z. Walczyk. *Vibration of Mechanical Systems in the Computer Approach* [in Polish], Part II, WNT, Warszawa, 1993.

[17] J. Kruszewski, E. Wittbrodt, Z. Walczyk. *Vibration of Mechanical Systems in the Computer Approach* [in Polish], Part II, WNT, Warszawa, 1993.

[18] S.H. Lee, S.S. Hsieh. Expedient implicit integration with adaptative time stepping algorithm for nonlinear transient analysis. *Comp. Meths. Appl. Mech. Engrg.* **81**: 151–172, 1990.

[19] L. Majewski, M. Trombski. Spatial model of a crane with clearances in the kinematic pairs of a telescopic jib [in Polish], *4th Pol. Conf. on Mechanics of Textile Machinery and Cranes*, Bielsko-Biała, June 1993, p. 195.

[20] W. Nowacki. *Theory of Elasticity* [in Polish], PWN, Warszawa, 1970.

[21] E. Rusiński. *Finite Element Method - the "Cosmos" System* [in Polish], MWKŁ, Warszawa, 1994.

[22] B.S. Sarma, T.K. Varadan, G. Prathap. On various formulations of large amplitude free vibrations of beams. *Computers & Structures* **29**(6): 959–966, 1988.

[23] W.I. Smirnow. *Higher Mathematics* (Polish translation), PWN, Warszawa, 1962.

[24] R. Solecki, J. Szymkiewicz. *Rod Systems and Surface Girders – Dynamic Analysis* [in Polish], Arkady, Warszawa, 1964.

[25] J. Storch, G. Strang. Paradox lost: natural boundary conditions in the Ritz-Galerkin method, *Int. J. Num. Meth. Engng.* **26**: 2255–2266, 1988.

[26] G. Strang, G.J. Fix. *An analysis of the finite element method.* Prentice-Hall, Inc. Englewood Cliffs, N.J. 1973.

[27] S. Timoshenko, J. Goodier. *Theory of Elasticity* (Polish translation), Arkady, Warszawa, 1962.

[28] O.C. Zienkiewicz, R. L. Taylor. *The Finite Element Method.* Fourth edition, 1991.

[29] O. C. Zienkiewicz, J. Z. Zhu. A simple error estimator and adaptative procedure for practical engineering analysis. *Int. J. Num. Meth. Engng.* **24**: 337–357, 1987.

[30] M. Życzkowski. *Strength of Structural Elements* [in Polish], PWN, Warszawa, 1988.