Computer Assisted Mechanics and Engineering Sciences, 6: 403–414, 1999.

Efficient free vibration analysis of large structures with close or multiple natural frequencies. Part I: Undamped structures

In-Won Lee, Hyung-Jo Jung and Man-Cheol Kim
Department of Civil Engineering
Korea Advanced Institute of Science and Technology
Taejon 305-701, Korea

(Received March 8, 1999)

An efficient numerical method which can calculate the eigenproblem for the large structural system with multiple or close natural frequencies is presented. The method is formulated by the accelerated Newton–Raphson method to the transformed problem. The method can calculate the natural frequencies and mode shapes without any numerical instability which may be encountered in the well-known methods such as the subspace iteration method or the determinant search method which has been widely used for solving eigenvalue problem. The efficiency of the method is verified by comparing convergence and solution time for numerical examples with those of the subspace iteration method and the determinant search method.

Keywords: free vibration analysis, accelerated Newton–Raphson method, multiple or close natural frequencies

1. Introduction

The analysis of structures under dynamic loads is considerable importance in many fields of engineering. If the dynamic analysis is performed by the mode superposition method, the eigenproblem must be first solved.

If structures with multiple or close natural frequencies, such as multi-span bridges, containment buildings of nuclear power plants, cable-stayed bridges, tires and the structures whose cross-sections are symmetric are analyzed by the subspace iteration method [2, 3, 5, 8, 9] or the determinant search method [2, 3, 4, 8] which has been mainly used for solving eigenproblems, the numerical instability or the slow convergence may be often encountered. What is mentioned above shows the serious shortcoming of the existing method.

Lee and Robinson [7] proposed an efficient solution method in the case of structures with multiple or close natural frequencies in order to improve numerical stability and increase convergence. To further improve the method, the accelerated Newton–Raphson method is proposed here. As examples for calculating multiple or close eigenvalues and the corresponding mode shapes, the simply-supported plate and the cooling tower structure are analyzed to prove the efficiency of the proposed method.

The objective of this paper is to present an efficient solution method in order to improve numerical stability and increase convergence in the case of structures with multiple or close natural frequencies. Numerical examples are presented to show the efficiency of the proposed method.

2. Method of analysis

2.1. Problem definition

Consider a generalized eigenvalue problem such as,

$$K\phi_j = \lambda_j M\phi_j \qquad (j = 1, 2, 3, \ldots, n), \tag{1}$$

$$\phi_j^T M\phi_j = 1\,, \tag{2}$$

where K and M are the stiffness matrix and mass matrix of order n, respectively. M is assumed to be positive definite and K positive semidefinite. λ_j is the jth natural frequency squared and ϕ_j the corresponding mode shape.

Let p eigenvalues, λ_i $(i = 1, 2, \ldots, p)$, be equal or close and the corresponding eigenvectors denoted by ϕ_i $(i = 1, 2, \ldots, p)$. The objective is to solve for the p multiple or close eigenvalues and associated eigenvectors.

The strategy for the eigenvalue problem of structures with multiple or close eigenpairs is to find the vectors x_i $(i = 1, \ldots, p)$ in the subspace of the eigenvectors ϕ_i $(i = 1, \ldots, p)$ and to rotate the vectors in the subspace to find the eigenvectors as shown in Fig. 1.

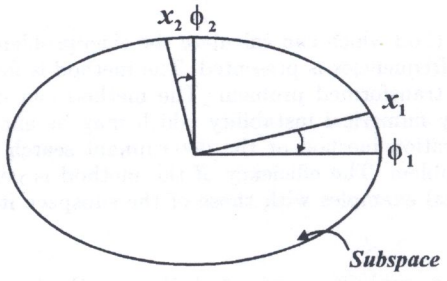

Fig. 1. Relation between Φ and vectors X in the subspace of Φ

Let $X = [x_1, \ldots, x_p]$ be the vectors in the subspace Φ, and X be the orthonormal with respect to M. Then

$$\Phi = XZ\,, \tag{3}$$

$$X^T MX = I_p\,, \tag{4}$$

where Z is the unknown rotation matrix of order p. Introducing Eq. (3) into Eq. (1), we get

$$KXZ = MXZ\Lambda\,. \tag{5}$$

Let

$$DZ = Z\Lambda\,, \tag{6}$$

where $D = [d_1, d_2, \ldots, d_p] = X^T KX$ and symmetric. Then,

$$KXZ = MXDZ \tag{7}$$

and

$$KX = MXD \tag{8}$$

or

$$Kx_i = MXd_i \qquad (i = 1, \ldots, p). \tag{9}$$

We obtain the p multiple or close eigenvalues and associated eigenvectors from Eqs. (3), (6) and (9).

Note when $\lambda_1 = \ldots = \lambda_p$, from Eq. (8)

$$D = \Lambda\,, \qquad X = \Phi\,.$$

2.2. Newton–Raphson method [7]

Let us assume that initial approximate solutions of Eq. (9), $d_i^{(0)}$ and $x_i^{(0)}$, are available. Denote an approximate eigenvalue and the corresponding eigenvector after k iterations by $d_i^{(k)}$ and $x_i^{(k)}$ $(k = 0, 1, 2, \ldots)$. Then, we have

$$r_i^{(k)} = r_i^{(k)}(d_i^{(k)}, x_i^{(k)}) = Kx_i^{(k)} - MX^{(k)}d_i^{(k)}\,, \tag{10}$$

$$(X^{(k)})^T MX^{(k)} = I_p\,, \tag{11}$$

where the residual vector, $r_i^{(k)}$, is not generally zero because of substitution of approximate values into Eq. (9). In order to make the residual vector a null vectors, the Newton–Raphson method is applied.

$$r_i^{(k+1)} = Kx_i^{(k+1)} - MX^{(k+1)}d_i^{(k+1)} = 0\,, \tag{12}$$

$$(X^{(k+1)})^T MX^{(k+1)} = I_p\,, \tag{13}$$

where

$$d_i^{(k+1)} = d_i^{(k)} + \Delta d_i^{(k)}\,, \tag{14}$$

$$x_i^{(k+1)} = x_i^{(k)} + \Delta x_i^{(k)}\,, \tag{15}$$

where $X^{(k+1)} = [x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_p^{(k+1)}]$, $\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$ are unknown incremental values of $d_i^{(k)}$ and $k_i^{(k)}$.

Substituting Eqs. (10), (11), (14) and (15) into Eqs. (12), (13) and neglecting the nonlinear terms, we obtain the linear simultaneous equations for $\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$:

$$-Kx_i^{(k)} + MX^{(k)}d_i^{(k)} = K\Delta x_i^{(k)} - M\Delta X^{(k)}d_i^{(k)} - MX^{(k)}\Delta d_i^{(k)}\,, \tag{16}$$

$$(X^{(k)})^T MX^{(k)} + 2(X^{(k)})^T M\Delta X^{(k)} = I_p\,. \tag{17}$$

If the λ_i $(i = 1, 2, \ldots, p)$ are multiple or close eigenvalues, the off-diagonal elements of D are zero or very small compared with its diagonal ones, thus the 2nd term in right hand side of Eq. (16) may be approximated by $d_{ii}^{(k)}M\Delta x_i^{(k)}$, yielding

$$-Kx_i^{(k)} + MX^{(k)}d_i^{(k)} = K\Delta x_i^{(k)} - d_{ii}^{k}M\Delta x^{(k)} - MX^{(k)}\Delta d_i^{(k)}\,, \tag{18}$$

$$(X^{(k)})^T M\Delta x_i^{(k)} = 0\,. \tag{19}$$

Writing Eqs. (18) and (19) in matrix form, we get

$$\begin{bmatrix} K - d_{ii}^{(k)}M & -MX^{(k)} \\ -(X^{(k)})^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{Bmatrix} = - \begin{Bmatrix} r_i^{(k)} \\ 0 \end{Bmatrix}. \tag{20}$$

The coefficient matrix for the incremental values is of order $n+p$, symmetric and nonsingular [7]. If the shift is near an eigenvalue, numerical sensitivity problems in the inverse iteration method with shift can be encountered. However, the Newton–Raphson method resolves the above problems, which is the main difference compared with the inverse iteration method with shift.

The above algorithm using the Newton–Raphson method, despite of its rapid convergence, is not efficient, since a new coefficient matrix has to be formed and refactorized in each iteration step.

2.3. Modified Newton–Raphson method [7]

The complete elimination procedure in each iteration may be avoided by using the modified Newton–Raphson method in Eq. (20) as follows :

$$\begin{bmatrix} K - d_{ii}^{(0)} M & -MX^{(k)} \\ -(X^{(k)})^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{Bmatrix} = - \begin{Bmatrix} r_i^{(k)} \\ 0 \end{Bmatrix}. \tag{21}$$

The coefficient matrix in Eq. (21) is both nonsingular and symmetric.

Convergence rates of $d_i^{(k+1)}$ and $x_i^{(k+1)}$ in Eq. (21) using the modified Newton–Raphson method can be written as

$$\gamma_i^{(k+1)} = h^2 \gamma_i^{(k)} \tag{22}$$

$$\theta_i^{(k+1)} = h \theta_i^{(k)} \tag{23}$$

where

$$\gamma_i^{(k+1)} = \left| \frac{d_{ii} - d_{ii}^{(k+1)}}{d_{ii}} \right|, \qquad h = \max_{i, i \neq j} \left| \frac{\lambda^* - d_{ii}^{(0)}}{d_{ii} - d_{ii}^{(0)}} \right|,$$

and $\lambda^* = \lambda_1 = \lambda_2 = \cdots = \lambda_p$. As shown in Eqs. (22) and (23), the convergence rate of eigenvalue, $\gamma_i^{(k+1)}$, is quadratic in h and that of eigenvector, $\theta_i^{(k+1)}$, linear in h.

Once the submatrix $(K - d_{ii}^{(0)} M)$ is decomposed into the LDL^T (L: lower triangular matrix, D: diagonal matrix), a small number of operations are required for the solution of Eq. (9) in the succeeding iterations, since the vector $Mx_i^{(k)}$ in the coefficient matrix is only changed in each iteration. However, due to negligence of the small nonlinear term $(d_{ii}^{(k+1)} - d_{ii}^{(0)}) M \Delta x_i^{(k)}$, the convergence is lower. Thus, the number of iterations for a solution increases. The above scheme has been presented by Lee and Robinson [7].

2.4. Accelerated Newton–Raphson method

To further improve the eigenvector, the accelerated scheme is proposed here, that is,

$$\begin{bmatrix} K - d_{ii}^{(0)} M & -MX^{(k)} \\ -(X^{(k)})^T M & 0 \end{bmatrix} \begin{Bmatrix} \Delta x_i^{(k)} \\ \Delta d_i^{(k)} \end{Bmatrix} = - \begin{Bmatrix} r_i^{(k)} \\ 0 \end{Bmatrix}, \tag{24}$$

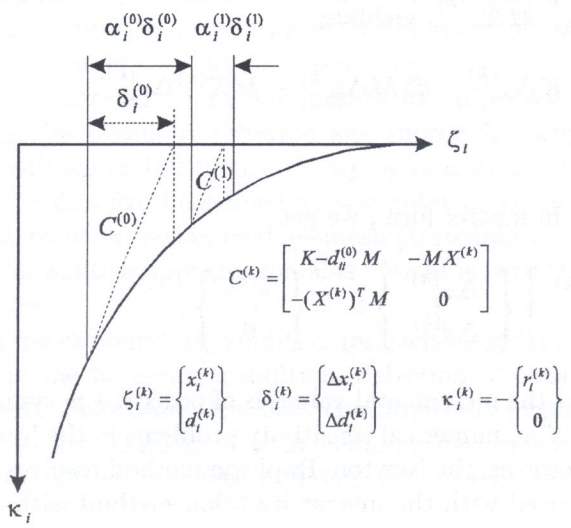

Fig. 2. The accelerated Newton–Raphson method

$$d_i^{(k+1)} = d_i^{(k)} + \Delta d_i^{(k)}, \tag{25}$$

$$x_i^{(k+1)} = x_i^{(k)} + \alpha_i^{(k)} \Delta x_i^{(k)}. \tag{26}$$

$\alpha_i^{(k)}$ is a step length to minimize the norm of the residual vector. It can be evaluated by using the least square technique as follows:

$$\frac{\partial}{\partial \alpha_i^{(k)}} \left\{ (r_i^{(k+1)})^T r_i^{(k+1)} \right\} = 0 \tag{27}$$

$$\alpha_i^{(k)} = -\frac{N_1 - N_2}{D} \tag{28}$$

where

$$N_1 = (\Delta x_i^{(k)})^T (K - d_{ii}^{(k+1)} M)(K - d_{ii}^{(k+1)} M) x_i^{(k)},$$

$$N_2 = (\Delta x_i^{(k)})^T (K - d_{ii}^{(k+1)} M) \left(\sum_{\substack{j=1, \\ j \neq i}}^{p} d_{ji}^{(k+1)} M x_j^{(k)} \right),$$

$$D = (\Delta x_i^{(k)})^T (K - d_{ii}^{(k+1)} M)(K - d_{ii}^{(k+1)} M) \Delta x_i^{(k)}.$$

Note that $d_{ii}^{(k+1)}$ and $\Delta x_i^{(k)}$ have been obtained by Eqs. (25) and (24), respectively.

2.5. Operation count and summary of algorithm

In order to obtain an estimate of the cost to solve an eigenvalue problem, consider the number of Central Processor operations required for solution. The actual cost includes, of course, the cost of the Peripheral Processor time. However, this time is system and programming dependent and is therefore not considered in this investigation.

Let one operation equal one multiplication which nearly always is followed by an addition. Assume that the half bandwidths of K and M, i.e. m_K and m_M, are full. A summary of the steps in the accelerated Newton–Raphson method together with the corresponding number of operations is given in the Table 1.

The number of operations for evaluating $\alpha_i^{(k+1)}$ in the first iteration step is $2nm_K + 2nm_M + (s+6)n$. This is large compared to $(p+4)nm_K + 2nm_M + (1/2)n(p^2+7p+4)$ which is required in each iteration step in the modified Newton–Raphson method. However, the number of $7n$ operations is more required all the way after the 2nd iteration, which is negligible, because we use computational results in the previous step. Thus, solution time of the proposed method decreases.

2.6. Numerical stability [7]

The most remarkable characteristic of the accelerated Newton–Raphson method is numerical stability. The numerical stability can be proved by identifying the nonsingularity of the coefficient matrix of Eq. (21).

Let the coefficient matrix of Eq. (21) be denoted by $C^{(k)}$, that is

$$C^{(k)} = \begin{bmatrix} K - d_{ii}^{(0)} M & -MX^k \\ -(X^{(k)})^T M & 0 \end{bmatrix}. \tag{29}$$

Table 1. Operation count for accelerated Newton–Raphson method

Calculation	Number of operations
$K - d_{ii}^{(0)}$	$n(m_M + 1)$
$LDL^T = K - d_{ii}^{(0)} M$	$\frac{1}{2} n m_K (m_K + 3)$
Iteration $k = 1, 2, 3, \ldots$	
$k = 1$	
$K x_i^{(k)}$	$n(2m_K + 1)$
$M x_i^{(k)} (i = 1, 2, \ldots, p)$	$pn(2m_M + 1)$
$k = 2, 3, \ldots$	
$K x_i^{(k)} = K(x_i^{(k-1)} + \alpha_i^{(k-1)} \Delta x_i^{(k-1)})$	n
$M x_i^{(k)} = M(x_i^{(k-1)} + \alpha_i^{(k-1)} \Delta x_i^{(k-1)})$	n
$r_i^{(k)} = K x_i^{(k)} - \sum_{j=1}^{p} d_{ji}^{(k)} M x_j^{(k)}$	pn
$LDL^T = MX^k$	$pn(m_K + \frac{1}{2}(p+1))$
Solve Eq. (21) for $\Delta d_i^{(k)}$ and $\Delta x_i^{(k)}$	$n(2m_K + p + 1)$
Solve Eq. (26) for $\alpha_i^{(k)}$	$2nm_K + 2nm_M + (p+5)n + 1$
$x_i^{(k+1)} = x_i^{(k)} + \alpha_i^{(k)} \Delta x_i^{(k)}$	n
$d_i^{(k+1)} = d_i^{(k)} + \Delta d_i^{(k)}$	0
Number of Operations for $k = 1$	$(p+6)nm_K + 2(p+1)nm_M + \frac{1}{2}n(p^2 + 7p + 18)$
Number of Operations for $k = 2, 3, \ldots$	$(p+4)nm_K + 2nm_M + \frac{1}{2}n(p^2 + 5p + 20)$

The determinant of $C^{(k)}$ is a continuous function of the approximate eigenvalue and eigenvectors. Hence, If $C^{(k)}$ is nonsingular when the approximate value in $C^{(k)}$ becomes the exact one, then it will be also nonsingular for close enough approximations. The resulting matrix C will be

$$C = \begin{bmatrix} K - \lambda_i M & -M\Phi \\ -\Phi^T M & 0 \end{bmatrix}, \qquad i = 1, 2, \ldots, p. \tag{30}$$

To prove the nonsingularity of C, we introduce the following eigenproblem

$$CY = M^* Y D \tag{31}$$

where

$$M^* = \begin{bmatrix} M & 0 \\ 0 & I_p \end{bmatrix}, \tag{32}$$

$$Y = [y_1 \; y_2 \; \cdots \; y_{n+p}], \tag{33}$$

$$D = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{n+p}). \tag{34}$$

The eigenpairs of the eigenproblem Eq. (29) are as follows

- Eigenvector y: $\begin{Bmatrix} \phi_i \\ e_i \end{Bmatrix}, \begin{Bmatrix} \phi_i \\ -\frac{1}{e_i} \end{Bmatrix}, \begin{Bmatrix} \phi_k \\ 0, \end{Bmatrix} \qquad i = 1, \ldots, p, \qquad k = p+1, \ldots, n$ (35)

- Eigenvalue μ: $e_{ii}, \; -\dfrac{1}{e_{ii}}, \; (\lambda_k - \lambda_i)$ (36)

where

$$e_i^T = (0, \ldots, 0, e_{ii}, 0, \ldots, 0),$$

$$e_{ii} = \frac{(\lambda_i - \lambda_j) + \sqrt{(\lambda_i - \lambda_j)^2 + 4}}{2}, \qquad i, j = 1, 2, \ldots, p.$$

Considering the determinant of Eq. (29)

$$\det[C] = \det[M^*] \det[D] = (-1)^p \det[M] \prod_{k=p+1}^{n} (\lambda_k - \lambda_i). \tag{37}$$

The determinant of C is not zero because of $\det[M] \neq 0$ by definition. The nonsingularity of the matrix $C^{(k)}$ is shown. That is, the numerical stability of the proposed method is proved.

2.7. Missed eigenvalues

Some of the eigenvalues and corresponding eigenvectors of interest may be missed when the initial approximations are not suitable. In order to check whether this occurs, the Sturm-sequence property [2] may be applied. A computed eigenvalue can be checked using the above property with negligible extra computation, since the decomposition of the matrix $(K - d_{ii}^{(0)})$ has already been carried out during the procedure for the solution of Eq. (21). If some of the eigenvalues of interest are detected to be missing, the solutions can be found by the accelerated Newton–Raphson method.

3. Numerical examples

The simply-supported plate and the cooling tower are analyzed to verify the efficiency of the proposed method. By using three methods separately, the subspace iteration method, the determinant search method and the proposed method, each convergence and solution time(CPU time) used to calculate 10 and 15 eigenpairs with error norm of 10^{-6} and 10^{-9} (20 eigenpairs in the case of the cooling tower) is compared, where the error norm [2] is computed by

$$\text{error norm} = \frac{\left\| K x_i^{(k)} - M X^{(k)} d_i^k \right\|_2}{\left\| K x_i^{(k)} \right\|_2}. \tag{38}$$

Especially to get the best results we applied the accelerated scheme [5] to the subspace iteration method. Intermediate results with relative error of 10^{-1} in the subspace iteration method are used as initial values of the proposed method. The relative error [2] in the subspace iteration method is computed as follows

$$\text{relative error} = \left| \frac{\lambda_i^{(k+1)} - \lambda_i^{(k)}}{\lambda_i^{(k+1)}} \right|. \tag{39}$$

$\alpha_i^{(k)}$ is applied to the eigenpair whose error norm is over 10^{-1}. All runs are executed in the IRIS4D-20-S17.

3.1. Simply-supported plate structure

The simply-supported plate structures shown in Fig. 3 consist of 36 nine-node elements, 169 nodes and 701 degrees o freedom. The stiffness matrix and mass matrix have the mean half-bandwidths of 89.

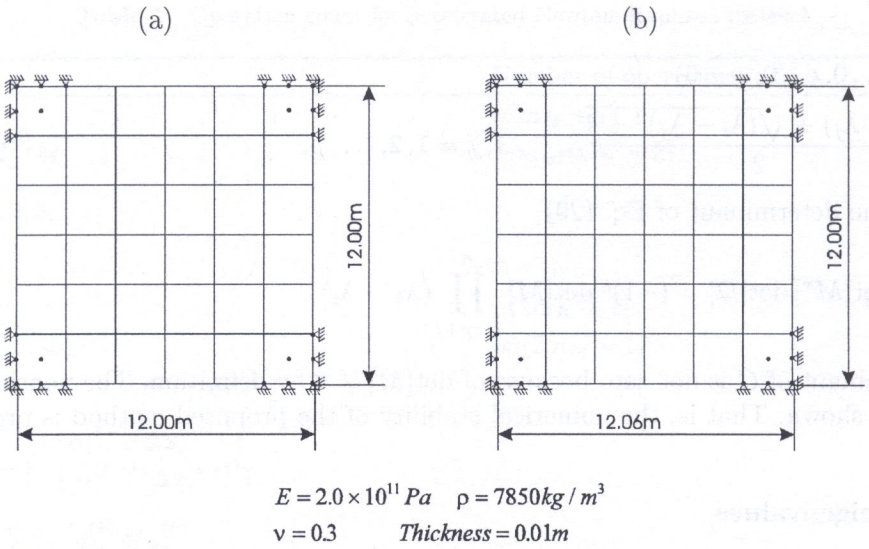

Fig. 3. Simply-supported plate structure [1]; (a) multiple natural frequency, (b) close natural frequency

3.1.1. Multiple natural frequencies

The simply-supported plate structure with multiple natural frequencies is shown in Fig. 3a. $\alpha_i^{(k)}$ is applied to the 6th, the 8th and the 10th eigenpair with error norm exceeding 10^{-1}.

Each solution time for three methods to have 10 and 15 eigenpairs with the error norm of 10^{-6} and 10^{-9} is summarized in Table 2, in which we check the solution time and the convergence rate especially in the case of 10 eigenpairs with error norm 10^{-9}. If we let the solution time for the proposed method be 1, it takes 1.6 times for the accelerated subspace iteration method, 6.5 times for the determinant search method. For each solution method the convergence of the 8th eigenpair to which $\alpha_i^{(k)}$ is applied is represented in Fig. 4. As shown in the above figure, we can see that the convergence of the proposed method is much superior to that of the accelerated subspace iteration and of the determinant search method.

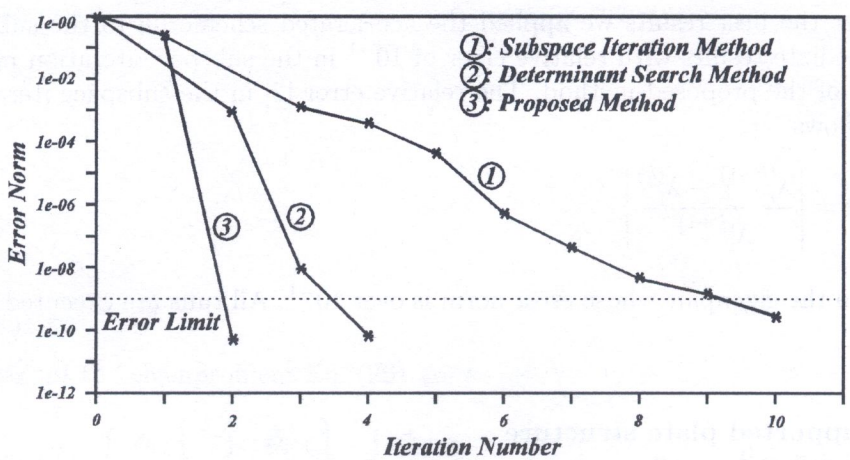

Fig. 4. Convergence of the 8th eigenpair

Table 2. Solution time (CPU time, sec) of plate with multiple natural frequencies

Analysis Method	p=10		p=15	
	EN=10^{-6}	EN=10^{-9}	EN=10^{-6}	EN=10^{-9}
Proposed Method	138.1 (1.0)	172.4 (1.0)	213.5 (1.0)	245.9 (1.0)
Acclerated Subspace Iteration Method	221.0 (1.6)	274.6 (1.6)	487.4 (2.3)	607.2 (2.5)
Determinant Search Method	1072.4 (7.8)	1118.6 (6.5)	1570.2 (7.4)	1618.8 (6.6)

EN = ErrorNorm

3.1.2. Close natural frequencies

The simply-supported plate structure with close natural frequencies is shown in Fig. 3b. $\alpha_i^{(k)}$ is applied to the 7th, the 8th and the 10th eigenpair with error norm exceeding 10^{-1}.

Each solution time for three methods to have 10 and 15 eigenpairs with the error norm of 10^{-6} and 10^{-9} is summarized in Table 3, in which we check the solution time and the convergence rate especially in the case of 10 eigenpairs with error norm 10^{-9}. If we let the solution time for the proposed method be 1, it takes 1.7 times for the accelerated subspace iteration method, 4.7 times

Table 3. Solution time (CPU time, sec) of plate with close natural frequencies

Analysis Method	p=10		p=15	
	EN=10^{-6}	EN=10^{-9}	EN=10^{-6}	EN=10^{-9}
Proposed Method	156.5 (1.0)	177.4 (1.0)	237.9 (1.0)	290.8 (1.0)
Acclerated Subspace Iteration Method	167.9 (1.1)	291.9 (1.7)	350.9 (1.5)	742.6 (2.6)
Determinant Search Method	813.2 (5.2)	832.1 (4.7)	1146.7 (4.8)	1194.1 (4.1)

EN = ErrorNorm

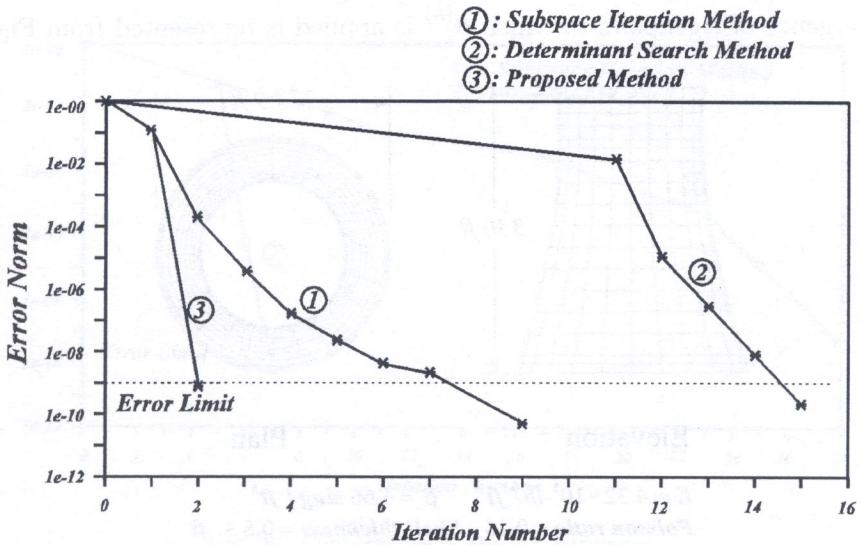

Fig. 5. Convergence of the 7th eigenpair

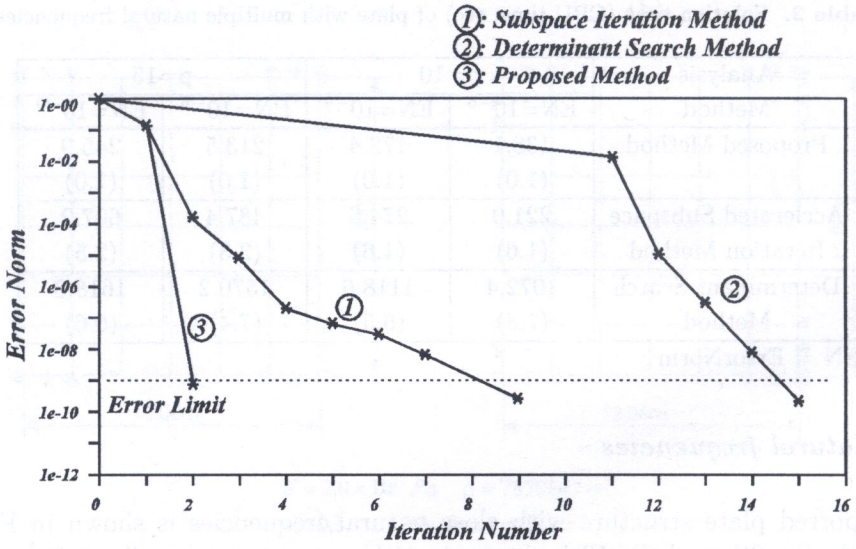

Fig. 6. Convergence of the 8th eigenpair

for the determinant search method. For each solution method the convergence of eigenpairs to which $\alpha_i^{(k)}$ is applied is represented from Fig. 5 to Fig. 6. As shown in the above figures, we can see that the convergence of the proposed method is superior to that of the accelerated subspace iteration and of the determinant search method.

3.2. Cooling tower structure

The cooling tower structure shown in Fig. 7 consists of 408 four-node shell elements, 432 nodes and 2,448 degrees-of-freedom. The stiffness and the mass matrix have the mean half-bandwidths of 201. $\alpha_i^{(k)}$ is applied to the 8th and the 10th eigenpair with error norm exceeding 10^{-1}.

Each solution time for two solution methods to get 10 and 20 eigenpairs with the error norm of 10^{-6} and 10^{-9} is summarized in Table 4, in which we check the solution time and the convergence rate especially in the case of 10 eigenpairs with error norm 10^{-9}. Determinant search method is not applied because it did not give us the good results. If we let the solution time for the proposed method be 1, it takes 2.0 times for the accelerated subspace iteration method. For each solution method the convergence of eigenpairs to which $\alpha_i^{(k)}$ is applied is represented from Fig. 8 to Fig. 9.

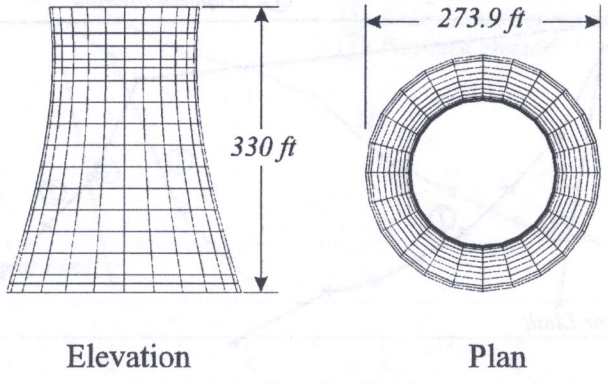

Fig. 7. Cooling tower structure [6]

Table 4. Solution time (CPU time, sec) of cooling tower

Analysis Method	p=10		p=15	
	EN=10^{-6}	EN=10^{-9}	EN=10^{-6}	EN=10^{-9}
Proposed Method	2785.8 (1.0)	3067.7 (1.0)	5104.2 (1.0)	5576.5 (1.0)
Acclerated Subspace Iteration Method	4854.9 (1.6)	6182.5 (2.0)	6383.6 (1.3)	15829.3 (2.8)
Determinant Search Method	No Sol.	No Sol.	No Sol.	No Sol.

EN = ErrorNorm

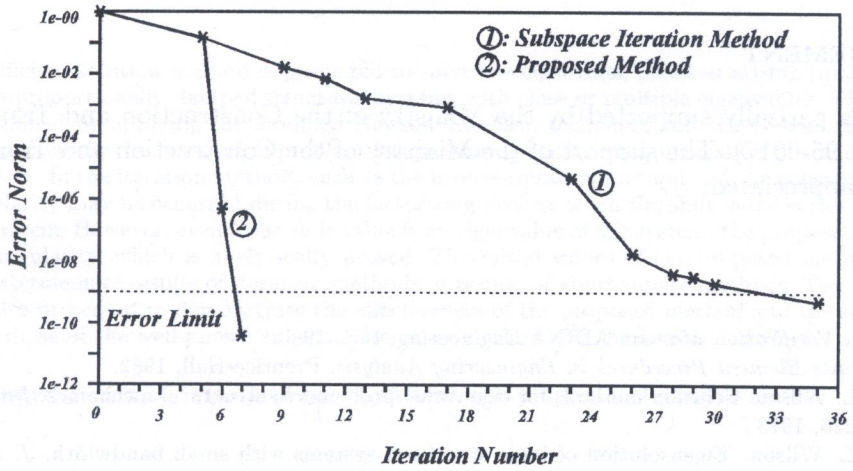

Fig. 8. Convergence of the 8th eigenpair

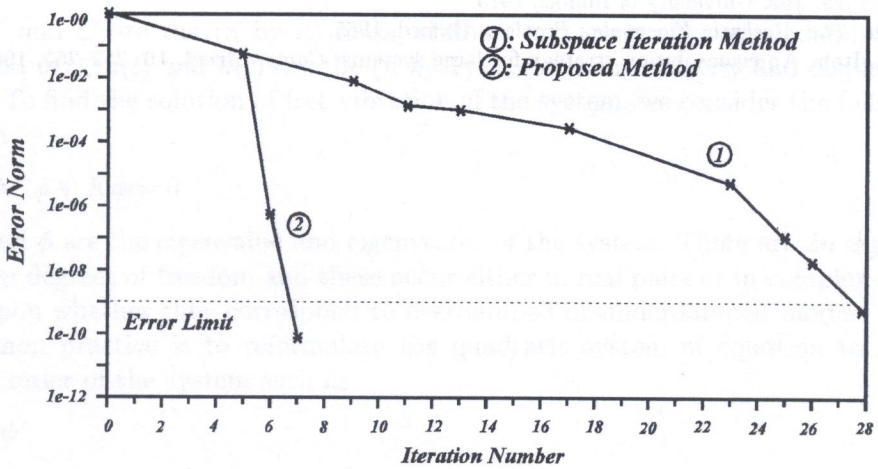

Fig. 9. Convergence of the 9th eigenpair

4. Conclusions

This paper proposes an efficient numerical method for the multiple or close eigenvalue problems using the accelerated Newton–Raphson method. As shown in the examples, the proposed method has the characteristics as follows.

1. The proposed method is a general technique which can compute the eigenpairs of a structure efficiently without any numerical instability in case of multiple or close eigenvalues as well as distinct eigenvalues.
2. The proposed method will not be affected by the eigenpairs previously calculated, because each eigenpair is essentially obtained independently.
3. The proposed method is simple and numerically stable, and converges very fast.
4. Missed eigenpairs can be detected with negligible operations in passing and can be found by the proposed method.

Acknowledgement

This research was partially supported by the Ministry of the Construction and Transportation of Korea (Grant no. 95-0017). The support of the Ministry of the Construction and Transportation of Korea is greatly appreciated.

References

[1] *ADINA System Verification Manual.* ADINA Engineering, Inc., 1983.
[2] K.J. Bathe. *Finite Element Procedures in Engineering Analysis.* Prentice-Hall, 1982.
[3] K.J. Bathe, E.L. Wilson. Solution methods for eigenvalue problems in structural mechanics. *Int. J. Num. Meth. Engrg.* **6**: 213–226, 1973.
[4] K.J. Bathe, E.L. Wilson. Eigensolution of large structural systems with small bandwidth. *J. Eng. Mech. Div.* **99**: 467–479, 1973.
[5] K.J. Bathe, S. Ramaswamy. An accelerated subspace iteration method. *Computer Methods in Appl. Mech. and Eng.* **23**: 313–331, 1980.
[6] A. Habibulah, E.L. Wilson. *SAP90 Sample Example and Verification Manual.* Computers and Structures, Inc., 1989.
[7] I.W. Lee, A.R. Robinson. Solution techniques for large eigenvalue problems in structural dynamics. *Structural Research Series No. 462.* University of Illinois, 1979.
[8] J.H. Wilkinson. *The Algebraic Eigenvalue Problem.* Oxford, 1965.
[9] E.L. Wilson, I. Itoh. An eigensolution strategy for large systems. *Comp. Struct.* **16**: 259–265, 1983.