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Numerical studies of dynamic stability under small random parametric excitations

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An efficient numerical procedure is proposed to obtain mean-square stability regions for both single-degree-of-freedom and two-degree-of-freedom linear systems under parametric bounded noise excitation. This procedure reduces the stability problem to a matrix eigenvalue problem. Using this approach, ranges of applicability to the well-known stochastic averaging method are discussed. Numerical results show that the small parameter size in the stochastic averaging method can have a significant effect on the stability regions. The influence of noise on the shape of simple and combination parametric resonances is studied.

Keywords: random vibration, stochastic averaging, mean square stability, bounded noise.

1. Introduction

The stochastic excitations arise in most real systems and their presence, generally is the rule. Very often they have small intensity and one can use models with small parameters for the mathematical descriptions of the systems. The most powerful method to deal with such models is the stochastic averaging method which goes back to book by Stratonovich [1].

Consider the following dynamical system in R^N:

$$\frac{dx_j(t)}{dt} = \varepsilon f_j(\mathbf{x}) + \sqrt{\varepsilon}\xi_r(t) g_{jr}(\mathbf{x}), \tag{1}$$

where $f_j(\cdot)$ and $g_{jr}(\cdot)$ are deterministic functions, $x_j(t)$ is the jth component of the vector $\mathbf{x}(t)$, ε is a small positive parameter, $\xi_r(t)$ is zero-mean stationary stochastic process with correlation function.

$$B_{rs}(t) = E[\xi_r(t+u)\xi_s(u)].$$

It is well-known (see, e.g. [2–4]) that for sufficiently small ε, and under some conditions, the solutions of the system (1) may be approximated by the solutions of the following Itô system:

$$dx_j(t) = \varepsilon a_j(\mathbf{x})dt + \varepsilon \sigma_{jk}(\mathbf{x}) dw_k(t), \tag{2}$$

where $w_k(t)$ are independent standard Wiener processes, a_j and σ_{jk} can be obtained as follows:

$$a_j = \left\langle f_j(\mathbf{x}(t)) + \int\limits_{-\infty}^{0} g_{is}(\mathbf{x}(t+\tau)) \frac{\partial}{\partial x_i} g_{jr}(\mathbf{x}(t)) B_{rs}(\tau) d\tau \right\rangle_T , \tag{3}$$

$$\sigma_{ji}\sigma_{ki} = \left\langle \int\limits_{-\infty}^{\infty} g_{ks}(\mathbf{x}(t+\tau)) g_{jr}(\mathbf{x}(t)) B_{rs}(\tau) d\tau \right\rangle_T , \tag{4}$$

where $\langle\cdot\rangle_T$ denotes a time-averaging operation,

$$\langle[\cdot]\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int\limits_0^T [\cdot]dt,$$

and $\sigma_{ji}\sigma_{ki}$ is the (j,k) element of the product of matrix σ and its transpose σ'. This result is one of the forms of the stochastic averaging method, and it has many extensions and applications (see, e.g. books [3, 5–7], and surveys [8–10]).

The method allows the system to be approximated by the simpler systems only over the finite time intervals. Nevertheless, it is often used in stability questions (see, e.g. [3, 5]). For example, in the case of the single-degree-of-freedom (SDOF) system

$$\frac{d^2x(t)}{dt^2} + 2\varepsilon\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2[1+\sqrt{\varepsilon}\xi(t)]x = 0, \tag{5}$$

the method gives the following approximate condition for the stability of the nth moment [3]:

$$\zeta > \frac{\pi}{8}(n+2)\omega_0 S(2\omega_0), \tag{6}$$

where $S(\omega)$ is the spectral density of excitation $\xi(t)$. However a considerable additional work is needed to make the approximation (2) valid in stability problems and caution is recommended in this case [11]. It should be noted that using the condition (6) in applications has stirred a controversy [12]. The quite important issue is obtaining a range of change ε, for which the stability of the averaged system implies the stability of the basic system. It is difficult to obtain estimations for this range, and it was analytically done for the system (5) only in the case of telegraphic noise excitation [13], i.e., for the Markov process $\xi(t) = \pm b$ with μ^{-1} being the mean time between jumps. The telegraphic noise has only two states and so its usage is very limited.

In this paper we consider the so-called bounded noise excitation model:

$$\xi(t) = \beta\sin[\theta + \gamma t + \alpha w(t)], \tag{7}$$

with $w(t)$ standing for the standard Wiener process, θ is an uniformly distributed in $[0, 2\pi]$ a random variable independent of $w(t)$; β, γ, α are deterministic parameters. This process is recently widely used as a random excitation in the dynamical systems (see, e.g., [14–18]). It is a zero mean stationary process with the following correlation function $B(t)$ and spectral density $S(\omega)$:

$$B(t) = \frac{\beta^2}{2}\exp\left\{-\frac{\alpha^2|t|}{2}\right\}\cos(\gamma t),$$

$$S(\omega) = \frac{1}{\pi}\int\limits_0^\infty e^{-i\omega t}B(t)dt = \frac{\alpha^2\beta^2(\omega^2+\gamma^2+\alpha^4/4)}{4\pi[(\omega^2-\alpha^4/4-\gamma^2)^2+\alpha^4\omega^2]}. \tag{8}$$

The density is considered as a good approximation to the well-known Dryden and von Karman spectra of wind turbulence by adjusting the values of the parameters β, α, γ [3]. We consider the mean square (asymptotic) stability for both, the system (5) and for the two-degree-of-freedom (TDOF) coupled linear systems with parametric random excitation (7). This notion of stability implies that all second moments of the system solutions tend to zero as $t\to\infty$, and it is important for many applications (see, e.g. discussions in [5]). An efficient numerical method is proposed to obtain the stability regions. This relies on the results of the paper [19] which allow to reduce the mean square stability investigations to some matrix eigenvalue problems.

The outline of the present paper is as follows. The mean square stability for SDOF and TDOF coupled linear systems with the random parametric excitation (7) is considered in Secs. 2 and 3 respectively. Here, the ranges of change ε for which the mean square stability of averaged system implies the stability of the basic system, are obtained numerically. It should be noted that the system (5) is not asymptotically stable if $\varepsilon = 0$ thus the very precise numerics are needed for obtaining the ranges. This method is quite efficient in studing simple and combination parametric resonances for TDOF coupled linear systems with parametric random excitation (7) as it is shown in Sec. 4.

2. Stability regions for SDOF system

Let us consider the system (5) with excitation (7). This system may represent, e.g. the lateral motion of a slender column under both axial and lateral excitations, when the motion is dominated by a single mode (see, e.g. [3] for details).The mean square stability for this system, in the case of telegraphic noise excitation, was considered in [20]. One can easily argue that the vector

$$\mathbf{y}(t) := \mathrm{col}\left\{\left(\frac{dx(t)}{dt}\right)^2,\ x(t)\frac{dx(t)}{dt},\ x^2(t)\right\}$$

satisfies the following equation in R^3:

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y} + \xi(t)\mathbf{C}\mathbf{y}, \qquad t > 0, \tag{9}$$

where

$$\mathbf{A} = \begin{pmatrix} -4\varepsilon\zeta\omega_0 & -2\omega_0^2 & 0 \\ 1 & -2\varepsilon\zeta\omega_0 & -\omega_0^2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & -2\sqrt{\varepsilon}\omega_0^2 & 0 \\ 0 & 0 & -\sqrt{\varepsilon}\omega_0^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us note that the solution of Eq. (9) is a functional of the Wiener process and therefore we should write $\mathbf{y}(t) = \mathbf{y}(t; w(s))$. Using the Cameron-Martin formula for the density of the Wiener measure under translation [21] we deduce that for all nonrandom λ,

$$E\left[\exp\{\mathrm{i}\lambda w(t)\}\mathbf{y}(t; w(s))\right] = \exp\left\{-\frac{\lambda^2 t}{2}\right\}E[\mathbf{y}(t; w(s) + \mathrm{i}\lambda s)],\ \mathrm{i} = \sqrt{-1}. \tag{10}$$

Since

$$\sin(\theta + \gamma t + \alpha w(t)) = \frac{\exp\{\mathrm{i}\theta + \mathrm{i}\gamma t + \mathrm{i}\alpha w(t)\} - \exp\{-\mathrm{i}\theta - \mathrm{i}\gamma t - \mathrm{i}\alpha w(t)\}}{2\mathrm{i}},$$

it follows from (9) and (10) that

$$\begin{aligned}\frac{dE[\mathbf{y}(t)]}{dt} = \mathbf{A}E[\mathbf{y}(t)] &+ \frac{\beta}{2\mathrm{i}}\exp\left\{\frac{-\alpha^2 t}{2} + \mathrm{i}\gamma t\right\}\mathbf{C}E[e^{\mathrm{i}\theta}\mathbf{y}(t; w(s) + \mathrm{i}\alpha s)] \\ &- \frac{\beta}{2\mathrm{i}}\exp\left\{\frac{-\alpha^2 t}{2} - \mathrm{i}\gamma t\right\}\mathbf{C}E[e^{-\mathrm{i}\theta}\mathbf{y}(t; w(s) - \mathrm{i}\alpha s)],\end{aligned}$$

$$\begin{aligned}\frac{dE[e^{\pm\mathrm{i}k\theta}\mathbf{y}(t; w(s) \pm \mathrm{i}k\alpha s)]}{dt} &= \mathbf{A}E[e^{\pm\mathrm{i}k\theta}\mathbf{y}(t; w(s) \pm \mathrm{i}k\alpha s)] \\ &+ \frac{\beta}{2\mathrm{i}}\mathbf{C}\exp\left\{\frac{-\alpha^2 t}{2} \mp k\alpha^2 t + \mathrm{i}\gamma t\right\}E[e^{\pm\mathrm{i}k\theta+\mathrm{i}\theta}\mathbf{y}(t; w(s) \pm \mathrm{i}k\alpha s + \mathrm{i}\alpha s)] \\ - \frac{\beta}{2\mathrm{i}}\mathbf{C}\exp\left\{\frac{-\alpha^2 t}{2} \pm k\alpha^2 t - \mathrm{i}\gamma t\right\}&E[e^{\pm\mathrm{i}k\theta-\mathrm{i}\theta}\mathbf{x}(t; w(s) \pm \mathrm{i}k\alpha s - \mathrm{i}\alpha s)], \quad k = 1, 2, 3, \ldots\ .\end{aligned} \tag{11}$$

With the introduction of functions

$$\mathbf{u}_k(t) := \frac{1}{(2\mathrm{i})^k} \exp\left\{ \mathrm{i}k\gamma - \frac{k^2\alpha^2 t}{2} \right\} E[e^{\mathrm{i}k\theta}\mathbf{y}(t; w(s) + \mathrm{i}k\alpha s)],$$

$$\mathbf{v}_k(t) := \frac{1}{(-2\mathrm{i})^k} \exp\left\{ -\mathrm{i}k\gamma - \frac{k^2\alpha^2 t}{2} \right\} E[e^{-\mathrm{i}k\theta}\mathbf{x}(t; w(s) - \mathrm{i}k\alpha s)], \qquad k = 1, 2, 3, \dots .$$

the infinite chain (11) becomes

$$\begin{aligned}
\frac{dE[\mathbf{y}(t)]}{dt} &= \mathbf{A}E[\mathbf{y}(t)] + \beta\mathbf{C}(\mathbf{u}_1(t) + \mathbf{v}_1(t)), \\
\frac{d\mathbf{u}_1}{dt} &= \left(-\frac{\alpha^2}{2} + \mathrm{i}\gamma \right) \mathbf{u}_1 + \mathbf{A}\mathbf{u}_1 + \lambda\mathbf{C}\mathbf{u}_2 + \frac{\beta}{4}\mathbf{C}E[\mathbf{y}(t)], \\
\frac{d\mathbf{v}_1}{dt} &= \left(-\frac{\alpha^2}{2} - \mathrm{i}\gamma \right) \mathbf{v}_1 + \mathbf{A}\mathbf{v}_1 + \lambda\mathbf{C}\mathbf{v}_2 + \frac{\gamma}{4}\mathbf{C}E[\mathbf{y}(t)], \qquad (12) \\
\frac{d\mathbf{u}_k}{dt} &= \left(-\frac{k^2\alpha^2}{2} + \mathrm{i}k\gamma \right) \mathbf{u}_k + \mathbf{A}\mathbf{u}_k + \lambda\mathbf{C}\mathbf{u}_{k+1} + \frac{\beta}{4}\mathbf{C}\mathbf{u}_{k-1}, \\
\frac{d\mathbf{v}_k}{dt} &= \left(-\frac{k^2\alpha^2}{2} - \mathrm{i}k\gamma \right) \mathbf{v}_k + \mathbf{A}\mathbf{v}_k + \beta\mathbf{C}\mathbf{v}_{k+1} + \frac{\beta}{4}\mathbf{C}\mathbf{v}_{k-1}, \qquad k = 1, 2, 3, \dots .
\end{aligned}$$

Note that if we have the cosine instead of the sine function in the excitation (7), then we obtain again the hierarchy (12), where

$$\mathbf{u}_k(t) := \frac{1}{2^k} \exp\left\{ \mathrm{i}k\gamma - \frac{k^2\alpha^2 t}{2} \right\} E[e^{\mathrm{i}k\theta}\mathbf{y}(t; w(s) + \mathrm{i}k\alpha s)],$$

$$\mathbf{v}_k(t) := \frac{1}{2^k} \exp\left\{ -\mathrm{i}k\gamma - \frac{k^2\alpha^2 t}{2} \right\} E[e^{-\mathrm{i}k\theta}\mathbf{y}(t; w(s) - \mathrm{i}k\alpha s)], \qquad k = 1, 2, 3, \dots .$$

Therefore, the mean square stability for the system (5) is reduced to the stability for infinite hierarchy (12). We have to close the hierarchy (12) and omitting the terms $\mathbf{u}_{n+1}$, $\mathbf{v}_{n+1}$ in the equations for $\mathbf{u}_n$, $\mathbf{v}_n$ is a common way. Then, the index n is called the truncation index. This procedure leads to the closed system of linear differential equations of first order with constant coefficients and is quickly convergent [19]. It is well-known that we have the asymptotic stability for this system, if and only, if the matrix of its coefficients has all eigenvalues with negative real parts. For sufficiently large truncation index, the asymptotic stability or instability of this system determines the mean-square stability or instability for the system (5). Thus, we obtain a tool for computing the stability curves that separate the stable and unstable regions for the system (5). Here we are mainly interested in the behavior of the curves for different ε.

In Fig. 1 these curves (solid lines) are presented in the parameter space (β, ζ) for $\omega_0 = 1$, $\gamma = 1$ and in Fig. 1a for $\alpha = 5$, $\varepsilon = 0.5$, in Fig. 1b for $\alpha = 0.5$, $\varepsilon = 0.001$. Also, here are presented the curves (dashed lines) obtained by the method of stochastic averaging, i.e. by using (6), (8). One can observe significant distinctions, but it follows from the computations that the appropriate curves coincide if $\varepsilon \leq 0.005$ and $\varepsilon \leq 0.0001$ respectively for $\alpha = 5$ and $\alpha = 0.5$. Computations of the eigenvalues were carried out over 1000×250 grid of equally spaced points ranging from $0 \leq \beta \leq 10$ to $0 \leq \zeta \leq 2.5$ with a help of the *Mathematica*. The truncation index is chosen in such a way that a further increase does not change the stability curve. In Table 1 we list the numerical results for the maximum real parts of the eigenvalues near the stability curve with truncation index $n = 6$ and $n = 100$. As it follows from the table, even the truncation index six gives very good results.

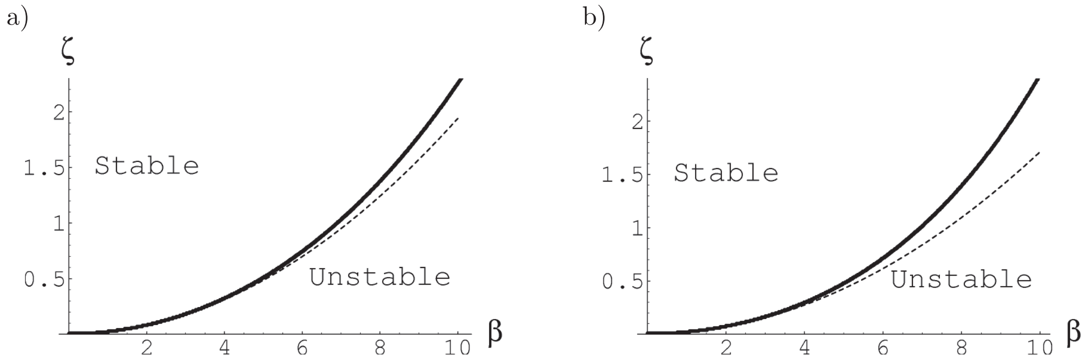

Fig. 1. The stability curves for Eq. (5) (solid lines) and the curves obtained from condition (6) (dashed lines) for the values $\omega_0 = 1$, $\gamma = 1$. In (a) $\alpha = 5$, $\varepsilon = 0.5$ and in (b) $\alpha = 0.5$, $\varepsilon = 0.001$.

Table 1. Numerical results for the maximum of real parts of the eigenvalues in the case $\gamma = 1$, $\omega_0 = 1$, $\alpha = 0.5$, $\varepsilon = 0.001$.

(β, ζ)	Truncation index	
	$n = 6$	$n = 100$
$(1, 0.018)$	-1.62337×10^{-6}	-1.62337×10^{-6}
$(3, 0.16)$	-4.27602×10^{-7}	-4.27602×10^{-7}
$(5, 0.473)$	-1.49448×10^{-6}	-1.49448×10^{-6}
$(7, 1.01)$	-7.77197×10^{-7}	-7.77196×10^{-7}
$(9, 1.856)$	-1.52326×10^{-6}	-1.52325×10^{-6}

3. Stability regions for TDOF system

We shall consider the TDOF dimensionless coupled linear system:

$$\begin{aligned}
&\frac{d^2x_1(t)}{dt^2} + 2\varepsilon\zeta_1\omega_1\frac{dx_1(t)}{dt} + \omega_1^2 x_1(t) - \sqrt{\varepsilon}\omega_1\omega_2\xi(t)x_2(t) = 0,\\
&\frac{d^2x_2(t)}{dt^2} + 2\varepsilon\zeta_2\omega_2\frac{dx_2(t)}{dt} + \omega_2^2 x_2(t) - \sqrt{\varepsilon}\omega_1\omega_2\xi(t)x_1(t) = 0.
\end{aligned} \tag{13}$$

This system may represent, e.g. motion of a beam of thin rectangular cross-section with stochastic follower force and its stability, considered in the paper [22] and the book [3]. Define:

$$\begin{aligned}
z_1(t) &= \left(\frac{dx_1(t)}{dt}\right)^2, & z_2(t) &= x_1(t)\frac{dx_1(t)}{dt},\\
z_3(t) &= \frac{dx_1(t)}{dt}\frac{dx_2(t)}{dt}, & z_4(t) &= \frac{dx_1(t)}{dt}x_2(t),\\
z_5(t) &= x_1^2(t), & z_6(t) &= x_1(t)\frac{dx_2(t)}{dt},\\
z_7(t) &= x_1(t)x_2(t), & z_8(t) &= \left(\frac{dx_2(t)}{dt}\right)^2,\\
z_9(t) &= x_2(t)\frac{dx_2(t)}{dt}, & z_{10}(t) &= x_2^2(t).
\end{aligned}$$

Then according to (13), these functions satisfy the following system of ten equations:

$$
\begin{aligned}
\frac{dz_1(t)}{dt} &= -4\varepsilon\zeta_1\omega_1 z_1(t) - 2\omega_1^2 z_2(t) + 2\sqrt{\varepsilon}\omega_1\omega_2\xi(t)z_4(t),\\
\frac{dz_2(t)}{dt} &= z_1(t) - 2\varepsilon\zeta_1\omega_1 z_2(t) - \omega_1^2 z_5(t) + \sqrt{\varepsilon}\omega_1\omega_2\xi(t)z_7(t),\\
\frac{dz_3(t)}{dt} &= -2\varepsilon(\zeta_1\omega_1 + \zeta_2\omega_2)z_3(t) - \omega_2^2 z_4(t) - \omega_1^2 z_6(t) + \sqrt{\varepsilon}\omega_1\omega_2\xi(t)[z_2(t) + z_9(t)],\\
\frac{dz_4(t)}{dt} &= z_3(t) - 2\varepsilon\zeta_1\omega_1 z_4(t) - \omega_1^2 z_7(t) + \sqrt{\varepsilon}\omega_1\omega_2\xi(t)z_{10}(t),\\
\frac{dz_5(t)}{dt} &= 2z_2(t),\\
\frac{dz_6(t)}{dt} &= z_3(t) - 2\varepsilon\zeta_2\omega_2 z_6(t) - \omega_2^2 z_7(t) + \sqrt{\varepsilon}\omega_1\omega_2\xi(t)z_5(t),\\
\frac{dz_7(t)}{dt} &= z_4(t) + z_6(t),\\
\frac{dz_8(t)}{dt} &= -4\varepsilon\zeta_2\omega_2 z_8(t) - 2\omega_2^2 z_9(t) + 2\sqrt{\varepsilon}\omega_1\omega_2\xi(t)z_6(t),\\
\frac{dz_9(t)}{dt} &= z_8(t) - 2\varepsilon\zeta_2\omega_2 z_9(t) - \omega_2^2 z_{10}(t) + \sqrt{\varepsilon}\omega_1\omega_2\xi(t)z_7(t),\\
\frac{dz_{10}(t)}{dt} &= z_9(t).
\end{aligned}
\tag{14}
$$

The vector

$$\mathbf{z}(t) := col\,(z_1(t), \ldots, z_{10}(t))$$

satisfies the Eq. (9) in R^{10} with 10×10 matrices A, C determined by the system (14). Therefore we can use the approach from the previous section to obtain the mean square stability curves for the system (13). The stochastic averaging method gives the approximate mean square stability conditions as follows [3]:

$$
\begin{aligned}
&\zeta_1 > \frac{\pi}{4}\omega_2 S^-, \qquad \zeta_2 > \frac{\pi}{4}\omega_1 S^-,\\
&\left(\zeta_1 - \frac{\pi}{4}\omega_2 S^-\right)\left(\zeta_2 - \frac{\pi}{4}\omega_1 S^-\right) > \frac{\pi^2}{16}\omega_1\omega_2 (S^+)^2,
\end{aligned}
\tag{15}
$$

where

$$S^{\pm} = S(\omega_1 + \omega_2) \pm S(\omega_1 - \omega_2).$$

In Fig. 2 the stability curves (solid lines) obtained from the numerical computations are presented together with the curves (dashed lines) obtained from the conditions (15) in the parametric space (ζ_2, ζ_1). Here, the parameters are $\omega_1 = 1$, $\omega_2 = 2$, $\gamma = 1$, $\beta = 4$ and in Fig. 1a, $\alpha = 1$, $\varepsilon = 0.01$, but in Fig. 1b, $\alpha = 5$, $\varepsilon = 0.5$. The appropriate curves coincide if $\varepsilon \leq 0.0001$ and $\varepsilon \leq 0.1$ for $\alpha = 1$ and $\alpha = 5$ respectively.

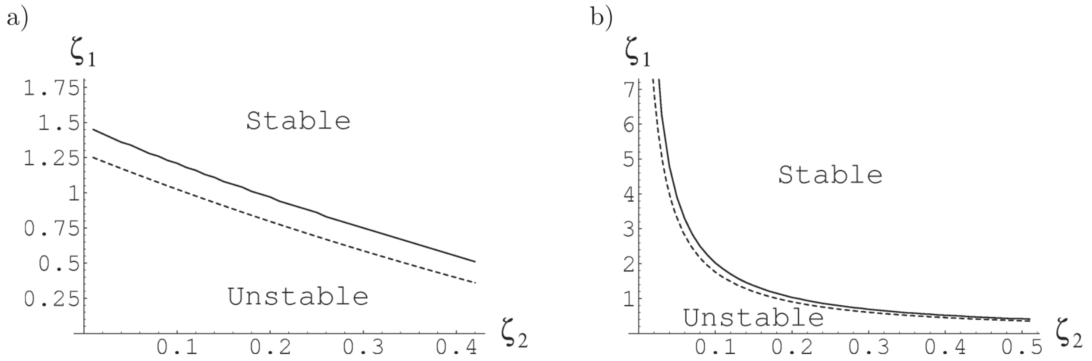

Fig. 2. The stability curve for Eqs. (13) (solid line) and the curve obtained from condition (15) (dashed line) for the values $\omega_1 = 1$, $\omega_2 = 2$, $\gamma = 1$, $\beta = 4$. In (a) $\alpha = 1$, $\varepsilon = 0.01$ and in (b) $\alpha = 5$, $\varepsilon = 0.5$.

Let us now consider the system (13) without assumption that the damping in the second equation is small. In that case the equation takes the form

$$\frac{d^2x_2(t)}{dt^2} + 2\zeta_2\omega_2\frac{dx_2(t)}{dt} + \omega_2^2 x_2(t) - \sqrt{\varepsilon}\omega_1\omega_2\xi(t)x_1(t) = 0, \qquad \zeta_2 < 1. \tag{16}$$

Using the stochastic averaging method, the following approximate stability condition was obtained in the book [3]:

$$\zeta_1 > \frac{\pi\omega_2}{4}(1-\zeta_2^2)^{-1/2}(S_d^+ - S_d^-), \tag{17}$$

where

$$S_d^{\pm} = \frac{1}{\pi}\int_0^{\infty} e^{-\zeta_2\omega_2\tau}\cos[(\omega_1 \pm \omega_{2d})\tau]B(\tau)d\tau,$$

$$\omega_{2d} = (1-\zeta_2^2)^{1/2}\omega_2.$$

In Fig. 3 the stability curve (solid line) obtained from the numerical computation is presented together with the curve (dashed line) obtained from the condition (17) in the parametric space

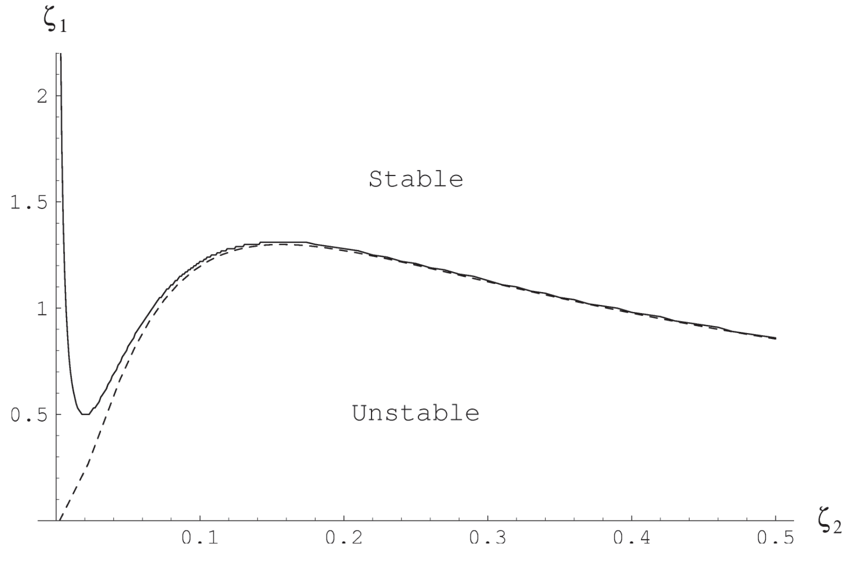

Fig. 3. The stability curve for Eqs. (13), (16) (solid line) and the curve obtained from condition (17) (dashed line) for the values $\omega_1 = 0.1$, $\omega_2 = 2$, $\gamma = 2$, $\beta = 10$, $\alpha = 0.5$, $\varepsilon = 0.00001$.

(ζ_2, ζ_1). Here the parameters are as follows: $\omega_1 = 0.1$, $\omega_2 = 2$, $\gamma = 2$, $\beta = 10$, $\alpha = 0.5$, $\varepsilon = 0.00001$. One can observe substantial distinction despite the fact that the parameter ε is quite small. It follows from the computations that the curves coincide if $\varepsilon \leq 0.0000001$. As in previous cases, the larger α leads to the larger ε in order to get the coincident curves.

4. Simple and combination parametric resonances

Let's consider again the system (13) where the frequencies are commensurable, i.e. $n_1\omega_1 + n_2\omega_2 = 0$ for some integer values of n_1 and n_2. It is well-known (see, e.g., [23, 24]) that, in this case, parametric resonances occur when the excitation $\xi(t)$ is in the form $\xi(t) = \beta \sin(\gamma t)$. The simple resonances arise when the frequency γ is near the values $2\omega_i/k$, $(i = 1, 2)$, while the combination resonances arise when γ is near the values $(\omega_1 + \omega_2)/k$, where k is a positive integer. In real applications a perfect periodicity may be rarely observed, and therefore the excitation model (7) can be quite interesting. In connection with this the influence of the random perturbation on the resonances is important. The numerical approach presented in previous sections is also efficient in this case. In Fig. 4 the stability curves in the parametric space (γ, β) are presented for the values $\omega_1 = 0.5$, $\omega_2 = 1$, $\zeta_1 = \zeta_2 = 0.01$, $\varepsilon = 1$; in Fig. 4a $\alpha = 0.1$ and in Fig. 4b $\alpha = 0.5$. One can observe from Fig. 4a the simple resonant tongues near the values 1 and 0.5 of the frequency γ. One can also clearly observe the combination resonant tongue near $\gamma = 1.5$ and the higher order resonance tongues. It follows from our computations that further increase of α leads to disappearances of the resonant tongues and the region of instability increase (Fig. 4b). However this increase leads also to the stabilization near the resonant frequency $\gamma = 1.5$.

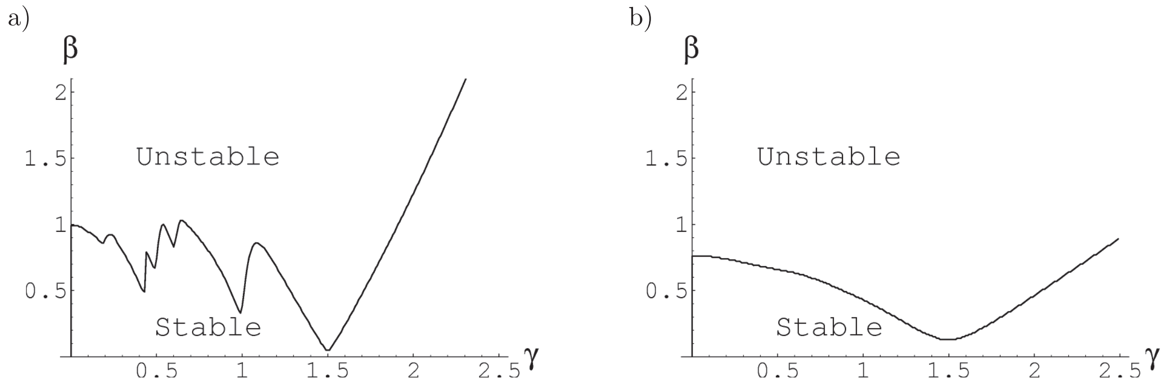

Fig. 4. The stability curves for Eqs. (13) for the values $\omega_1 = 0.5$, $\omega_2 = 1$, $\zeta_1 = \zeta_2 = 0.01$, $\varepsilon = 1$. In (a) $\alpha = 0.1$ and in (b) $\alpha = 0.5$.

The numerical procedure is quite efficient as one can observe from Table 2, where we list the numerical results for the maximum real parts of the eigenvalues near the stability curve at the resonant frequencies for the truncation indexes $n = 7$ and $n = 100$.

Table 2. Numerical results for the maximum of real parts of the eigenvalues in the case $\omega_1 = 0.5$, $\omega_2 = 1$, $\zeta_1 = \zeta_2 = 0.01$, $\varepsilon = 1$, $\alpha = 0.1$.

(γ, β)	Truncation index	
	$n = 7$	$n = 100$
$(0.5, 0.75)$	-0.000437452	-0.000435079
$(1, 0.38)$	-0.000560318	-0.000560318
$(1.5, 0.04)$	-0.00198176	-0.00198176

5. Conclusions

In this paper, an efficient numerical method for, of investigation of the effect of parametric random excitation on the mean square stability of SDOF and TDOF linear systems is proposed. The influence of the excitation is modelled using a bounded noise process. The method is based on reduction of the stability problem to the matrix eigenvalue problem. The efficiency of applications of the well-known stochastic averaging method to the stability problem is discussed. The results have shown how important is a proper choice of the range of small parameter in the stochastic averaging method, especially, if the excitation is narrow-band (α is small). The stability diagrams for TDOF system with the parametric random excitation are presented where simple and combination parametric resonances are observed.

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