Computer Assisted Mechanics and Engineering Sciences, 7: 625–639, 2000.

Solution sets for systems of linear interval equations

Zenon Kulpa and Karol Rosłaniec
Institute of Fundamental Technological Research, Polish Academy of Sciences
ul. Świętokrzyska 21, 00-049 Warsaw, Poland

(Received November 3, 1999)

The paper discusses various classes of solution sets for *linear interval systems of equations*, and their properties. Interval methods constitute an important mathematical and computational tool for modelling real-world systems (especially mechanical) with (bounded) uncertainties of parameters, and for controlling rounding errors in computations. They are in principle much simpler than general probabilistic or fuzzy set formulation, while in the same time they conform very well with many practical situations. Linear interval systems constitute an important subclass of such interval models, still in the process of continuous development. Two important problems in this area are discussed in more detail—the classification of so-called *united solution sets*, and the problem of overestimation of interval enclosures (in the context of linear systems of equations called also a *matrix coefficient dependence problem*).

1. Introduction

Interval computation methods, starting from an early paper by Warmus [26], through a series of monographs and textbooks [1, 4, 16, 17], grew into a well-established field providing mathematical and computational tools for modelling systems with uncertainties, see e.g. [14], and for fully controlling rounding errors in computation. In this approach, an uncertain (real) number is represented by an *interval* (a connected bounded subset) of real numbers which presumably contains the unknown exact value of the number in question. Thus, the uncertainty is bounded by the size of the interval; in addition, no commitment to a particular probability distribution (or its estimate) of the various alternative values within the interval needs to be made. This model is thus in principle much simpler than general probabilistic or fuzzy set formulations of uncertainty. Despite its simplicity, it conforms very well with many practical situations, like tolerance handling in mechanics or managing rounding errors in numerical computations. Also, so recently popular *fuzzy set* approach uses more and more often the interval formulations and methods. First, intervals can be considered as a special kind of a *fuzzy set membership function* (a "square-wave" membership function). Second, the so-called α-cut approach [3, 25] to handle more complex membership functions is based on replacing a fuzzy set problem with a set of interval problems: every interval problem is obtained by thresholding the original fuzzy set membership function at some value α of the function, $0 \le \alpha \le 1$.

The problems of modeling uncertainty and of reliable computing are also very important issues in computational analysis of mechanical structures, hence the importance of interval computation methods in this application area becomes recently widely recognized. This is signified, among others, by the growing number of publications [7, 14, 15, 19, 25]. The basic, and very important subfield of computational methods for mechanical applications concerns analysis of *linear mechanical systems* with parameter uncertainties. Here, the mathematical model of the problem is usually formulated as a *system of linear equations*. In the interval formulation, the coefficients of the system matrix and right-hand side are intervals, and hence we have the problem of solving a *system of interval linear equations*. The general theory of solving such systems is considerably advanced, see [5, 6, 14, 17, 18, 20, 21, 22, 23, 24], although the demands of practical applications still uncover new gaps in the theory and in development of practical computational algorithms, see e.g. the discussion of some of these problems in [14]. One set of problems here comes from the fact that the very notion of *solution* to a system of interval linear equations is rather far from being obvious, or at least is much more complicated than for the ordinary, noninterval linear systems. Depending on the purpose or application, different definitions of the solution to such a system are useful. In most of the interesting cases, we get *sets of solutions*, which come in a large variety of shapes, often quite complex and hard to calculate or characterize exactly. Therefore, from a practical point of view, it is crucial to work out comprehensive understanding of both the possible variety of definitions and shapes of these solution sets, and the variety of available practical methods for finding tractable approximations to them. Little work on characterization of shapes of these solution sets has been done. Some basic findings are scattered over various sources, e.g. [5, 14, 22], while seemingly little work has been devoted exclusively to that problem [2]. This paper aims to be another step in this general direction.

The paper starts with a brief introduction to interval arithmetic and systems of linear interval equations. Then, the basic types of solutions to these systems are presented. The most important in today's practical applications is the so-called *united solution set* (USS), hence the main part of the paper is devoted to it. We show diagrammatically the basic construction of the USS, various approximations to it, as well as a basic classification (and certain computational properties) of the diverse shapes of such sets. Next, the general problem of large practical importance, especially in mechanical applications, namely the overestimation problem of interval enclosures is discussed (in the context of linear interval systems it is usually called a *matrix coefficient dependence* problem). The discussion leads to the formulation of a so-called *parametric solution set* and its special cases for linear and symmetric coefficient dependence.

As mentioned above, a great deal of the results reported in the paper is presented or illustrated graphically. The graphical examples shown are all (at most) two-dimensional; however, in most cases the generalization to more dimensions is quite straightforward. Graphical (or, more precisely, diagrammatic) methods gain popularity recently, both for presenting and for discovering scientific facts [9], including facts about interval arithmetic [10, 11, 12, 13]. The classification of shapes and other properties of interval solution sets seems to be especially well suited for handling by diagrammatic methods. They offer more comprehensible characterization and understanding of the entities and relations in this domain, hence their adoption in this investigation by the authors.

2. Intervals and interval arithmetic

The notation used in this paper in general follows the standard conventions of interval literature, see e.g. [1, 4, 16, 17], with some minor modifications and additions. Modifications consist of wider use of operator notation instead of functional notation (in order to minimise the number of superfluous parentheses in formulas). Additions constitute the introduction of the "circumflex" symbol for interval radius, as well as an occasional use of the shorthand "±" constructor, and alternative variants of symbols for the TSS and CSS solution sets (see Sec. 3).

Usually, an interval is defined as a *pair of* (comparable) *elements* of some (at least *partially*) *ordered set*. In this paper we consider only *real intervals*, i.e. intervals defined over the set of real numbers $\mathbb{R}$. Also, for our purposes we can identify an interval with the set of elements lying between its endpoints (including the endpoints). A *proper real interval* x (or an *interval* x if no confusion arises) is thus defined as a subset of the set of real numbers $\mathbb{R}$ such that

$$x = [\underline{x}, \overline{x}] = \{\tilde{x} \in \mathbb{R} \,|\, \underline{x} \leq \tilde{x} \leq \overline{x}\},$$

where $\underline{x} \leq \overline{x}$, and $\underline{x} = \inf x$, $\overline{x} = \sup x$ are *endpoints* of the interval x. By $\tilde{x}$ we shall denote any element of the interval x. The set of all (real) intervals is denoted by $\mathbb{IR}$ and called a *(real) interval space*. The interval is called *thick* if $\underline{x} < \overline{x}$, and *thin* (or *point*) interval if $\underline{x} = \overline{x}$. For most purposes thin intervals can be identified with corresponding real numbers.

For an interval x the *midpoint*, *radius*, *magnitude* (or *absolute value*) and *mignitude* are defined, respectively, as follows,

$$\begin{aligned}
\check{x} &= \operatorname{mid} x = (\overline{x} + \underline{x})/2, \\
\hat{x} &= \operatorname{rad} x = (\overline{x} - \underline{x})/2, \\
|x| &= \operatorname{mag} x = \max(|\underline{x}|, |\overline{x}|), \\
\langle x \rangle &= \operatorname{mig} x = \min(|\underline{x}|, |\overline{x}|) \quad \text{if } 0 \notin x, \quad \langle x \rangle = 0 \ \text{ otherwise.}
\end{aligned}$$

Interval can be also expressed in terms of its midpoint and radius—this is a so-called *centered formulation*,

$$x = [\check{x} - \hat{x}, \check{x} + \hat{x}] = \check{x} + \Lambda \hat{x} = \check{x} \pm \hat{x}, \qquad \text{where } \Lambda = [-1, +1].$$

The notation using the Λ interval was introduced by Warmus as early as in 1956 [26] but it is not in general use today; the "$\pm$" notation we introduce in this paper as a convenient shorthand.

All operations and functions defined on reals may be naturally extended to cover interval operands according to the general formula

$$\begin{aligned}
x \diamond y &= \{\tilde{x} \diamond \tilde{y} \mid \tilde{x} \in x, \tilde{y} \in y\}, \\
f(x_1, x_2, \ldots, x_n) &= \{f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \mid \tilde{x}_i \in x_i, \ i = 1, 2, \ldots, n\}.
\end{aligned} \tag{1}$$

For many operations, including a standard arithmetic operations of *addition*, *subtraction*, *multiplication* and *division*, the resulting set is also an interval that can be conveniently defined in terms of endpoints (or midpoint and radius) of the argument intervals,

$$\begin{aligned}
x + y &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}] = (\check{x} + \check{y}) \pm (\hat{x} + \hat{y}), \\
-x &= [-\overline{x}, -\underline{x}] = -\check{x} \pm \hat{x}, \\
x - y &= x + (-y) = [\underline{x} - \overline{y}, \overline{x} - \underline{y}] = (\check{x} - \check{y}) \pm (\hat{x} + \hat{y}), \\
x \cdot y &= [\min(\underline{x}\underline{y}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\overline{y}), \max(\underline{x}\underline{y}, \overline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\overline{y})], \\
x / y &= x \cdot [1/\overline{y}, 1/\underline{y}] = x \cdot y / \underline{y}\overline{y} = x \cdot y / (\check{y}^2 - \hat{y}^2) \quad \text{if } 0 \notin y, \quad \text{undefined otherwise.}
\end{aligned} \tag{2}$$

As can be seen from the definitions, subtraction and division of intervals are not the inverse operations to addition an multiplication, respectively, differently than for the corresponding operations on reals. Other differences exist, most notably the *distributive law* $a(b+c) = ab + ac$ does not hold in general in interval arithmetic. Instead, we have the weaker *subdistributive law*,

$$x(y + z) \subseteq xy + xz.$$

For any bounded set of real numbers s we can define a *smallest interval enclosure* of the set, called also (interval) *hull* of the set

$$\operatorname{hull} s = [\inf s, \sup s].$$

E.g., for a two-element set $\{\alpha, \beta\}$ of real numbers we have $\operatorname{hull}\{\alpha, \beta\} = [\min(\alpha, \beta), \max(\alpha, \beta)]$. Thus, when an application of some operation or function produces a set which is not an interval, the hull of the set can be taken if there is a need to stay within interval arithmetic all the time (which is usually the case). Hence, Eqs. (1) are in practice used in the form

$$\begin{aligned}
x \diamond y &= \operatorname{hull}\{\tilde{x} \diamond \tilde{y} \mid \tilde{x} \in x, \ \tilde{y} \in y\}, \\
f(x_1, x_2, \ldots, x_n) &= \operatorname{hull}\{f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \mid \tilde{x}_i \in x_i, \ i = 1, 2, \ldots, n\}.
\end{aligned} \tag{3}$$

Most operations on intervals can be extended to interval matrices, by applying them componentwise to all matrix elements. In particular, *infimum*, *supremum*, *midpoint*, *radius*, *magnitude*, *intersection*, *addition* and *subtraction* are so defined. An interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times m}$ can be also considered as a set of real matrices, or as a matrix interval

$$\mathbf{A} = \left\{ \tilde{\mathbf{A}} \in \mathbb{R}^{n\times m} \mid \underline{\mathbf{A}} \leq \tilde{\mathbf{A}} \leq \overline{\mathbf{A}} \right\} = [\underline{\mathbf{A}}, \overline{\mathbf{A}}] = \check{\mathbf{A}} \pm \hat{\mathbf{A}}; \qquad \underline{\mathbf{A}}, \overline{\mathbf{A}}, \check{\mathbf{A}}, \hat{\mathbf{A}} \in \mathbb{R}^{n\times m}; \quad \hat{\mathbf{A}} \geq 0.$$

Matrix multiplication is defined as for real matrices, with the hull operation used as the final step, since the set $\{\tilde{\mathbf{A}}\tilde{\mathbf{B}} \mid \tilde{\mathbf{A}} \in \mathbf{A}, \tilde{\mathbf{B}} \in \mathbf{B}\}$ in general may not be an interval matrix. It is also important to remember that multiplication of interval matrices (contrary to the non-interval matrices and scalar intervals) is not associative, thus in general $\mathbf{A}(\mathbf{BC}) \neq (\mathbf{AB})\mathbf{C}$, unless $\mathbf{A}$ and $\mathbf{C}$ are thin (i.e., real) matrices.

The *boundary* (or the *vertex set*) of an interval matrix is a set of real matrices consisting of 2^t elements, where t is the number of thick interval coefficients of $\mathbf{A}$, and is defined as

$$\text{vert}\,\mathbf{A} = \left\{ \tilde{\mathbf{A}} \in \mathbf{A} \mid \tilde{a}_{ij} \in \text{vert}\, a_{ij} \right\} = \left\{ \tilde{\mathbf{A}} \in \mathbf{A} \mid \tilde{a}_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\} \right\} \tag{4}$$

Of course, for an interval x we have $\text{vert}\, x = \{\underline{x}, \overline{x}\}$.

3. Systems of linear interval equations

Let us consider a linear interval system of equations with an interval coefficient matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and an interval right-hand side vector $\mathbf{b} \in \mathbb{IR}^{n}$,

$$\mathbf{Ax} = \mathbf{b}. \tag{5}$$

Such an equation can be considered either as a system of interval equations, or as a set of real systems of equations. To these two interpretations correspond different concepts of a solution to this equation. For the first interpretation, the most natural definition leads to a so-called *algebraic solution*, which is simply an interval vector $\mathbf{x}$ that fulfills the above interval equation. However the algebraic solution does not always exist, and is of little use in many practical applications of interval equations. In most applications it is needed to look for all possible real vectors $\tilde{\mathbf{x}}$ that fulfill the set of equations of the form $\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, where $\tilde{\mathbf{A}} \in \mathbf{A}$ and $\tilde{\mathbf{b}} \in \mathbf{b}$ are all possible combinations of real matrices and right-hand side vectors contained in the interval matrices $\mathbf{A}$ and $\mathbf{b}$. Such a definition also parallels the natural extension of numerical operations and functions into the realm of intervals according to Eq. (1). Now, the $\mathbf{x}$ we are looking for is not a single interval vector, but rather a set of real vectors, not necessarily constituting a real interval. Therefore, it should be rather called a solution set—more specifically in this case, a *united solution set* (USS, for short). It is defined as

$$\Sigma(\mathbf{A}, \mathbf{b}) = \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists\tilde{\mathbf{A}} \in \mathbf{A})\ (\exists\tilde{\mathbf{b}} \in \mathbf{b})\ \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}\} = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid \mathbf{A}\tilde{\mathbf{x}} \cap \mathbf{b} \neq \emptyset\}. \tag{6}$$

In certain applications, two other kinds of solution sets happen to be useful [23, 24]. These are:

– a *tolerable solution set* (TSS),

$$\Sigma_{\subseteq}(\mathbf{A}, \mathbf{b}) = \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\forall\tilde{\mathbf{A}} \in \mathbf{A})\ (\exists\tilde{\mathbf{b}} \in \mathbf{b})\ \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}\} = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid \mathbf{A}\tilde{\mathbf{x}} \subseteq \mathbf{b}\}, \tag{7}$$

– a *controllable solution set* (CSS),

$$\Sigma_{\supseteq}(\mathbf{A}, \mathbf{b}) = \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\forall\tilde{\mathbf{b}} \in \mathbf{b})\ (\exists\tilde{\mathbf{A}} \in \mathbf{A})\ \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}\} = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid \mathbf{A}\tilde{\mathbf{x}} \supseteq \mathbf{b}\}. \tag{8}$$

Both TSS and CSS usually are not interval vectors, similarly as for the case of USS. A graphic illustration of the solution sets (6), (7) and (8) is shown in Fig. 1.

It is easy to see from the above definitions that always

$$\Sigma_{\subseteq}(\mathbf{A}, \mathbf{b}) \subseteq \Sigma(\mathbf{A}, \mathbf{b}) \qquad \text{and} \qquad \Sigma_{\supseteq}(\mathbf{A}, \mathbf{b}) \subseteq \Sigma(\mathbf{A}, \mathbf{b}).$$

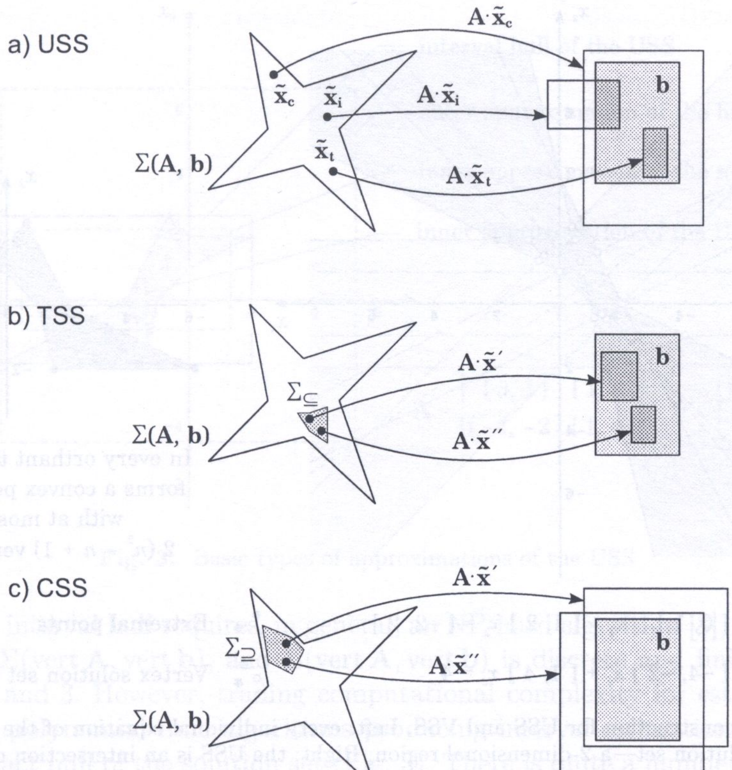

Fig. 1. Three kinds of solution sets for systems of linear interval equations: united solution set (a), tolerable solution set (b) and controllable solution set (c)

Other important relations hold between the above solution sets. E.g., it was proven that TSS and CSS cannot both be nonempty for a given nonsingular system of equations (except in the trivial case when both TSS and CSS are equal and are actually a real vector). Hence, Fig. 1a is not fully correct unless the set $\Sigma(\mathbf{A}, \mathbf{b})$ is unbounded.

Other possible solution sets will be introduced in the sequel. However, only the united solution set USS and its various approximations and characterizations will be considered in more detail.

4. United solution set

Usually, the set $\Sigma(\mathbf{A}, \mathbf{b})$ is not an interval vector, contrary to what may seem by analogy to the system of real equations. It is of rather complicated shape (in general, not necessarily convex, connected, or bounded). It is connected and bounded if the matrix $\mathbf{A}$ is regular. In this case, it constitutes an n-dimensional polyhedron which is a sum of at most 2^n convex polyhedrons obtained as intersections of the set $\Sigma(\mathbf{A}, \mathbf{b})$ with every of the 2^n orthants of the solution space $Ox_1 \ldots x_n$ [5]. The *convex hull* conv $\Sigma(\mathbf{A}, \mathbf{b})$ of this set is a minimal convex polyhedron containing $\Sigma(\mathbf{A}, \mathbf{b})$; as can be easily seen, the vertices of the convex hull constitute a subset of vertices of the solution set.

Figure 2 shows an example of the diagrammatic construction and representation of the USS, in the form developed by the first author of this paper (see also [13]). It shows in detail the structure of the system of equations and constitutes a basis for further analysis of various properties and types of solution sets (the details will be published separately). Every of the constituting interval equations (here 2-dimensional) may be represented by a sub-region of the 2-dimensional solution space Ox_1x_2. Solid lines represent so-called *boundary lines*, i.e., solution sets of all real equations obtained from the interval equation by taking all combinations of interval endpoints. Dashed lines correspond to solution sets for interval midpoints of the coefficients (they will be called *midlines* in the sequel). The solution set of the system of equations is the intersection of the sets corresponding

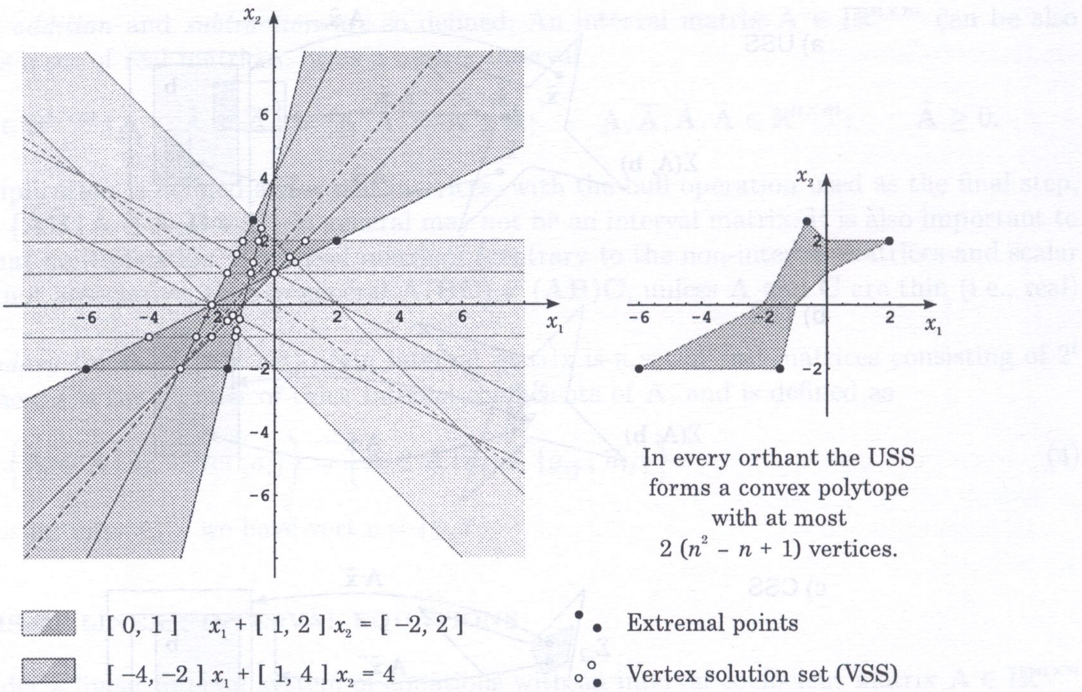

Fig. 2. Graphical construction for USS and VSS. Left: every individual equation of the system of equations produces its own solution set—a 2-dimensional region; Right: the USS is an intersection of these solution sets for individual equations

to individual equations (it is also shown separately at the right). The intersection of the central lines corresponds to the solution of the real system of equations for the midpoints of interval coefficients; obviously, it is always contained in the USS.

4.1. Vertex solution set

The black and white points in Fig. 2 indicate a *vertex solution set* (VSS), i.e. the set of solutions of all real systems of equations whose coefficients are all possible combinations of endpoints of the coefficients of the matrix **A** and vector **b**,

$$\Sigma(\operatorname{vert}\mathbf{A}, \operatorname{vert}\mathbf{b}) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists\tilde{\mathbf{E}} \in \operatorname{vert}\mathbf{A})(\exists\tilde{\mathbf{e}} \in \operatorname{vert}\mathbf{b})\tilde{\mathbf{E}}\tilde{\mathbf{x}} = \tilde{\mathbf{e}}\}.$$

They constitute intersections of boundary lines taken from different individual equations of the system. The VSS is a discrete and finite set of points, with number of elements equal to $2^t \leq 2^{n^2+n}$, where $t = t_{\mathbf{A}} + t_{\mathbf{b}}$ is the number of thick intervals in matrices **A** and **b** of the system.

Black points in Fig. 2 indicate *extremal points* of the USS, i.e. vertices of the *convex hull* $\operatorname{conv}\Sigma(\mathbf{A}, \mathbf{b})$ of the USS. Of course, they all belong to VSS. Note, however, that not all vertices of the USS must belong to the VSS (two such vertices can be seen in Fig. 2).

4.2. Approximations of the USS

Calculating (and representing) the exact solution set $\Sigma(\mathbf{A}, \mathbf{b})$ may be quite hard and impractical, especially for larger n. Therefore, for many practical purposes we are satisfied with various approximations to this set. The natural approximation is the *interval enclosure* of the set. The smallest (tightest) enclosure is the *hull* of the set, see e.g. [17],

$$\operatorname{hull}\Sigma(\mathbf{A}, \mathbf{b}) = [\inf\Sigma(\mathbf{A}, \mathbf{b}), \sup\Sigma(\mathbf{A}, \mathbf{b})].$$

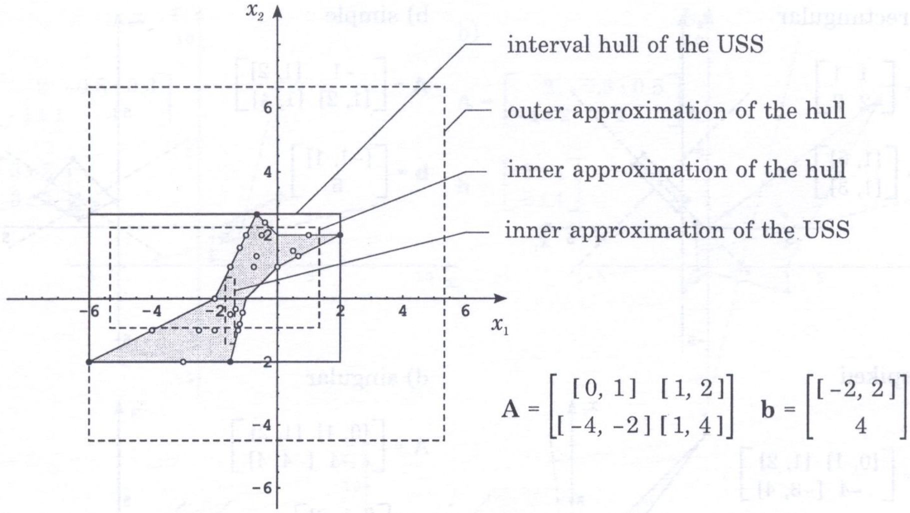

Fig. 3. Basic types of approximations of the USS

Calculation of an interval hull requires, in general, an NP-hard algorithm [8], despite the fact that $\text{hull}\,\Sigma(\mathbf{A},\mathbf{b}) = \text{hull}\,\Sigma(\text{vert}\,\mathbf{A}, \text{vert}\,\mathbf{b})$, and $\Sigma(\text{vert}\,\mathbf{A}, \text{vert}\,\mathbf{b})$ is discrete and finite; see the previous Section and Figs. 2 and 3. However, trading computational complexity for estimation accuracy, it is possible to derive polynomial-time algorithms producing interval estimations that are larger (or smaller) than the exact hull of the solution set (Fig. 3). There is quite a number of such algorithms, of various complexity, accuracy, applicability conditions and effectiveness, scattered over a wide selection of literature sources, e.g. [5, 7, 8, 14, 17, 19, 20, 21, 22, 23, 24]. Some of them are adequately developed and tested, but the other still require much research. Of much practical importance would be an appropriate systematization of them, leading to eventual creation of a knowledge base system giving advice as to which algorithm is best (e.g., fastest or most reliable) for any given system of interval equations.

4.3. Basic types of united solution sets

The variety of shapes of united solution sets is considerable. Fortunately, they can be classified into a few basic types, according to the preliminary classification scheme proposed here. In Fig. 4 we present the most basic types (for $n = 2$). The examples shown here (and in Sec. 4.4) are all due to the authors (see also [14]). As in Fig. 2, the dashed lines are midlines for component equations of the system. One of the simplest type is a *rectangular* solution set. This type of solution set occurs, e.g., for a real matrix **A** with orthogonal rows. Despite its simplicity, in general case there is no polynomial-time algorithm to find the hull of it—we need to solve at least 2^n systems of real equations, so it is of exponential complexity.

In the second example, the whole solutioni set is contained in one orthant (Fig. 4b). Although the shape is more complex than the previous one (the matrices contain more interval coefficients, edges are not parallel), there are known polynomial-time algorithms to find its hull. This is due to the knowledge that the solution set is contained in one orthant only. When the rectangular solution set is known to be confined to one orthant, its hull can be found in polynomial time as well.

The next type, a *spiked* set (Fig. 4c) is probably the most interesting. This shape characterizes systems near singularity—an analogue of ill-conditioned real systems. The more "spiky" the shape, the nearer is some real singular matrix to the matrix interval **A** of the system. Usually, then, widening (some) intervals in **A** leads to an even more spiky shape, as it usually brings the matrix **A** nearer to singularity.

The solution sets for singular matrices (Fig. 4d, e) produce an unbounded region in the $Ox_1 \ldots x_n$ space. Note how the singular system in Fig. 4d is obtained from that in Fig. 4c by changing only

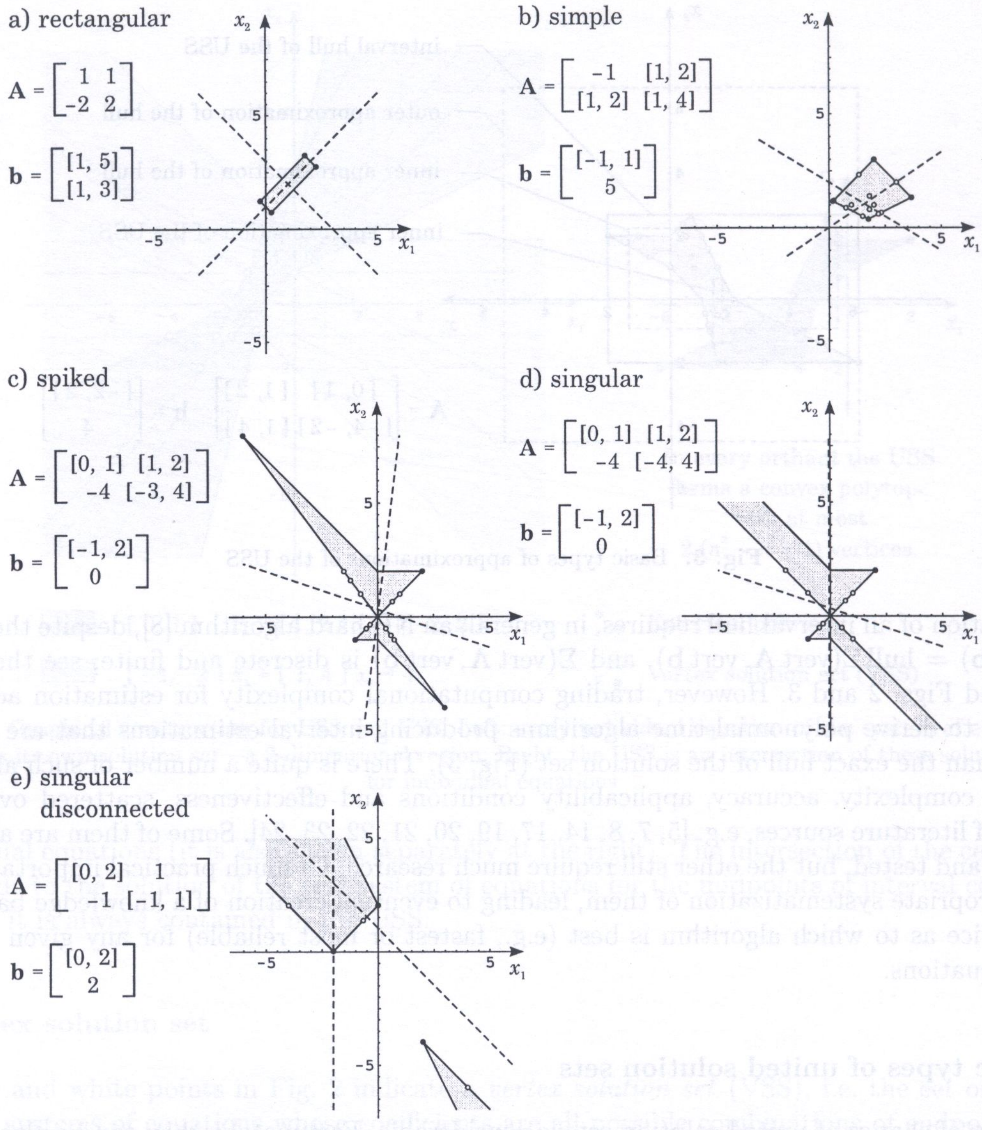

Fig. 4. Examples of some proposed basic shapes of united solution sets: *rectangular*—intervals only in **b**, while **A** is real with orthogonal rows (a); *simple*—the whole solution set confined to one orthant (b); *spiked*—with long spikes, usually due to large interval widths (c); *singular*—when spikes go to infinity (d); *singular disconnected*—with two disconnected components (more components possible for $n > 2$) (e)

the lower bound $\underline{a}_{22}$ of the coefficient a_{22} from -3 to -4. When $\underline{a}_{22}$ moves down from -3, the spikes in Fig. 4c become longer and longer until finally they go to infinity (with the sides of the spikes becoming parallel for $\underline{a}_{22} = -4$), when the matrix **A** becomes singular. For certain singular matrices, the solution set may become even disconnected (Fig. 4e). Exact conditions for that to occur were not yet published.

The appropriate classification of solution set shapes has direct application for fitting appropriate solution method to the given system of equations (i.e., given interval matrix **A** of the system), as was indicated above.

4.4. Dependence of the USS on interval width

In many applications, it is required to analyze the behaviour of solutions when the degree of uncertainty of data is varied. That translates to varying the widths of the interval coefficients of the system. Hence, investigation of how the solution set depends on the width of interval coefficients

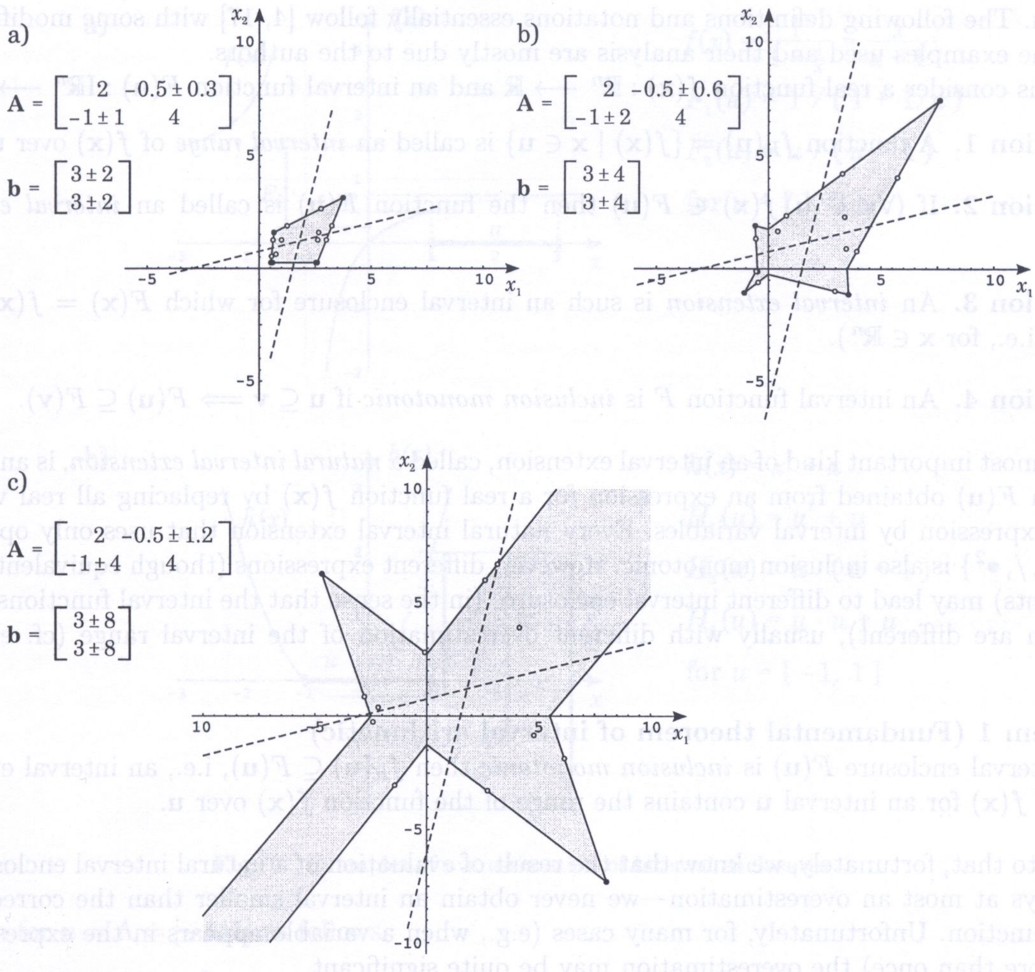

Fig. 5. Dependence of the USS on widths of interval coefficients. The example illustrates how much the solution type might change by scaling widths of coefficients (the last system is singular!)

of matrix **A** and vector **b** becomes of great significance. Figure 5 presents an example. First, we solved an exemplary system of interval equations obtaining a simple solution set (Fig. 5a). However, the system with doubled widths of coefficients (Fig. 5b) gives a quite different type of solution set—a spiked one. Again, a system of equations with coefficient widths doubled once more leads to a singular system with an unbounded solution set (Fig. 5c).

The example indicates that greater uncertainty of data (or parameters of the real-world system modelled by an interval system of equations) may lead to interval problems that are less tractable or even unbounded. That may reflect adequately the properties of the modelled system (e.g., of a truss that becomes a mechanism). However, in some cases that may be a result of overestimation effects of interval arithmetic. These effects, and how to avoid them, are discussed in the next three sections of the paper.

5. Overestimation problem for interval enclosures

As it was mentioned before, certain formal properties of interval arithmetic, like that the usual distributive law is not valid (only the weaker subdistributive law holds), may lead to overestimation errors during calculation of interval expressions. Indeed, if not taken properly into account, these effects may severely diminish the accuracy of interval estimates of solutions to linear systems of even moderate complexity. In this and the next sections, we attempt to explain the effects in more detail, show how they affect the accuracy of interval estimations, and discuss possible remedies to the problem. The following definitions and notations essentially follow [4, 17] with some modifications, while the examples used and their analysis are mostly due to the authors.

Let us consider a real function $f(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}$ and an interval function $F(\mathbf{u}) : \mathbb{IR}^n \longrightarrow \mathbb{IR}$.

Definition 1. A function $f_R(\mathbf{u}) = \{f(\mathbf{x}) \mid \mathbf{x} \in \mathbf{u}\}$ is called an *interval range* of $f(\mathbf{x})$ over $\mathbf{u} \in \mathbb{IR}^n$.

Definition 2. If $(\forall \mathbf{x} \in \mathbf{u})\, f(\mathbf{x}) \in F(\mathbf{u})$ then the function $F(\mathbf{u})$ is called an *interval enclosure* of $f(\mathbf{x})$.

Definition 3. An *interval extension* is such an interval enclosure for which $F(\mathbf{x}) = f(\mathbf{x})$ for all thin $\mathbf{x}$ (i.e., for $\mathbf{x} \in \mathbb{R}^n$).

Definition 4. An interval function F is *inclusion monotonic* if $\mathbf{u} \subseteq \mathbf{v} \Longrightarrow F(\mathbf{u}) \subseteq F(\mathbf{v})$.

The most important kind of an interval extension, called a *natural interval extension*, is an interval function $F(\mathbf{u})$ obtained from an expression for a real function $f(\mathbf{x})$ by replacing all real variables in the expression by interval variables. Every natural interval extension that uses only operations $\{+, -, \cdot, /, \bullet^2\}$ is also inclusion monotonic. However, different expressions (though equivalent for real arguments) may lead to different interval enclosures (in the sense that the interval functions defined by them are different), usually with different overestimation of the interval range (cf. examples below).

Theorem 1 (Fundamental theorem of interval arithmetic)
If an interval enclosure $F(\mathbf{u})$ is *inclusion monotonic* then $f_R(\mathbf{u}) \subseteq F(\mathbf{u})$, i.e., an interval enclosure $F(\mathbf{u})$ of $f(\mathbf{x})$ for an interval $\mathbf{u}$ contains the range of the function $f(\mathbf{x})$ over $\mathbf{u}$.

Due to that, fortunately, we know that the result of evaluation of a natural interval enclosure will be always at most an overestimation—we never obtain an interval smaller than the correct range of the function. Unfortunately, for many cases (e.g., when a variable appears in the expression for $f(\mathbf{x})$ more than once) the overestimation may be quite significant.

Two simple, one-dimensional examples are shown in Fig. 6. First, consider the expression $f_1(x) = 1/(1+1/x)$. In standard arithmetic, for $x \neq 0$, it is equivalent to a simpler formula $f_2(x) = x/(x+1)$, requiring only one division instead of two required for $f_1(x)$, but containing the variable x twice. That is, both expressions $f_1(x)$ and $f_2(x)$ define the same real function $f(x)$ of real variable x. However, natural interval extensions $F_1(u)$ and $F_2(u)$ are not equivalent. Indeed, putting $u = [1, 3]$ and calculating $F_1(u)$ and $F_2(u)$ using interval arithmetic rules, one gets

$$F_1([1,3]) = 1/(1 + 1/[1,3]) = 1/(1 + [1/3, 1]) = 1/[4/3, 2] = [1/2, 3/4],$$

which is equal to the range $f_R([1, 3])$, whereas

$$F_2([1,3]) = [1,3]/([1,3] + 1) = [1,3]/[2,4] = [1,3] \cdot [1/4, 1/2] = [1/4, 3/2].$$

Thus, $F_2([1, 3]) \neq F_1([1, 3]) = f_R([1, 3])$, and calculation according to the seemingly simpler formula $F_2(u)$ overestimates significantly the value of $f_R(u)$. However, we have still $f_R(u) \subseteq F_2(u)$, as prescribed by Theorem 1 (Fig. 6a).

The problem is that many functions cannot be expressed in the form in which the variables occur only once (or in certain other forms for which the overestimation problem does not arise [1]).

The second example (Fig. 6b) shows a function $h(x) = x^2 + x$ for which there is no arithmetic expression containing the variable x only once. Hence, all its natural interval extensions overestimate the range $h_R(u)$ for some interval arguments u. E.g., the three natural interval extensions for $h(x)$,

$$H_1(u) = u^2 + u,$$
$$H_2(u) = u \cdot (u + 1),$$
$$H_3(u) = u \cdot u + u,$$

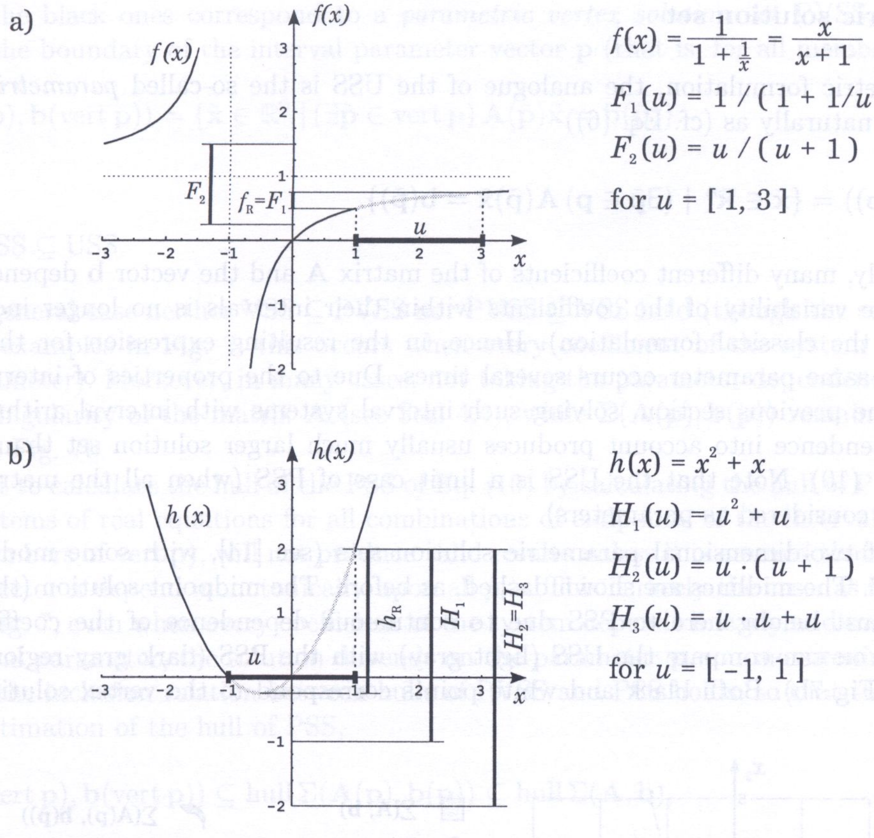

Fig. 6. Examples of overestimation of interval enclosures

evaluate for $u = \Lambda = [-1, 1]$ as follows,

$$H_1([-1,1]) = [-1,1]^2 + [-1,1] = [0,1] + [-1,1] = [-1,2],$$
$$H_2([-1,1]) = [-1,1] \cdot ([-1,1] + 1) = [-1,1] \cdot [0,2] = [-2,2],$$
$$H_3([-1,1]) = [-1,1] \cdot [-1,1] + [-1,1] = [-1,1] + [-1,1] = [-2,2] = H_2([-1,1]).$$

Note that when evaluating u^2 in H_1 we used the generic definition of an extension of real functions to interval operands, according to Eq. (1). Otherwise we would have got an overestimation of the value of u^2 that would in effect change H_1 into H_3. Since $h_R([-1,1]) = [-1/4, 2]$, see Fig. 6b, then every of the interval extensions H_1, H_2, H_3 overestimates the range in this case. However, the inclusion property according to Theorem 1 holds for all of them, as expected.

6. Coefficient dependence and parametric solutions

When one applies an interval formulation to some real-life system with uncertainties (e.g., to a linear mechanical problem), the interval coefficients of the resulting matrices of the system of equations usually depend, in often complex ways, on several uncertain (that is, interval) parameters of the original system. Realization of this situation leads to the so-called *parametric formulation* for the system of interval equations [6, 14, 22]. Let coefficients of a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ be functions of some vector of interval parameters $\mathbf{p}$. The parametric formulation of the system of linear interval equations is then given as

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{p}), \qquad \text{with} \quad \mathbf{p} = (p_1, p_2, \ldots, p_k), \tag{9}$$

where p_i, $i = 1, \ldots, k$, are given parameters varying over specified intervals and for every $\tilde{\mathbf{p}} \in \mathbf{p}$, $\mathbf{A}(\tilde{\mathbf{p}})$ and $\mathbf{b}(\tilde{\mathbf{p}})$ are real matrices.

6.1. Parametric solution set

For the parametric formulation, the analogue of the USS is the so-called *parametric solution set* (PSS), defined naturally as (cf. Eq. (6))

$$\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p})) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists \tilde{\mathbf{p}} \in \mathbf{p})\, \mathbf{A}(\tilde{\mathbf{p}})\tilde{\mathbf{x}} = \mathbf{b}(\tilde{\mathbf{p}})\}. \tag{10}$$

Since, usually, many different coefficients of the matrix **A** and the vector **b** depend on the same parameters, the variability of the coefficients within their intervals is no longer independent (as is assumed in the classical formulation). Hence, in the resulting expression for the (analytical) solution **x** the same parameter occurs several times. Due to the properties of interval arithmetic explained in the previous section, solving such interval systems with interval arithmetic without taking the dependence into account produces usually much larger solution set than the true one defined by Eq. (10). Note that the USS is a limit case of PSS (when all the matrix coefficients themselves are considered as parameters).

Examples of two-dimensional parametric solution sets (see [14], with some modifications) are shown in Fig. 7. The midlines are shown dashed, as before. The midpoint solution (the intersection of midlines) must belong here to PSS due to continuous dependence of the coefficients on the parameter **p**. You can compare the USS (light gray) with the PSS (dark gray region in Fig 7a, a curved line in Fig 7b). Both black and white points correspond to the vertex solution set (of the

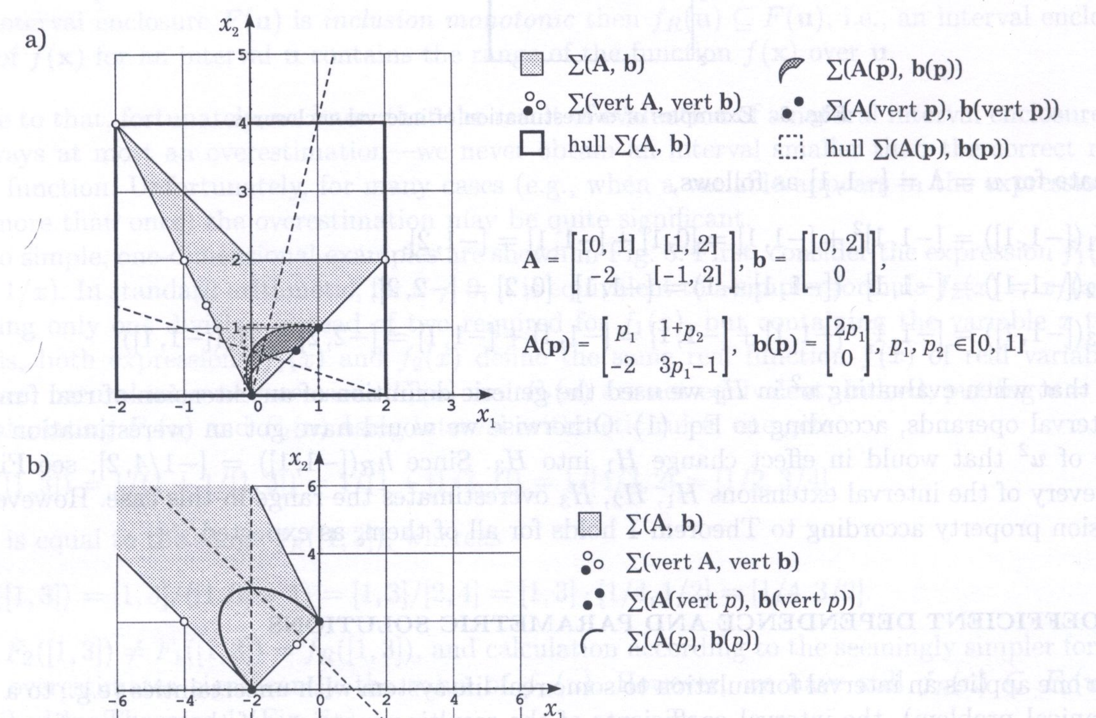

Fig. 7. Parametric solution set is usually much smaller than the corresponding USS, also for singular systems, where it can be bounded as well (b)

USS), while the black ones correspond to a *parametric vertex solution set* PVSS, i.e. the set of solutions for the boundary of the interval parameter vector $\mathbf{p}$ (that is, for all members of vert $\mathbf{p}$),

$$\Sigma(\mathbf{A}(\text{vert}\,\mathbf{p}), \mathbf{b}(\text{vert}\,\mathbf{p})) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists \tilde{\mathbf{p}} \in \text{vert}\,\mathbf{p})\, \mathbf{A}(\tilde{\mathbf{p}})\tilde{\mathbf{x}} = \mathbf{b}(\tilde{\mathbf{p}})\}.$$

Generally

$$\text{PVSS} \subseteq \text{PSS} \subseteq \text{USS}.$$

Note that in general case neither VSS $\subseteq$ PVSS nor PVSS $\subseteq$ VSS hold (though the second inclusion holds for the examples in Fig. 7: this occurs when every coefficient of the system depends on at most one parameter). Moreover, in many cases, not taking the parameter dependence into account may lead to singularity of the matrix $\mathbf{A}$ (see Sec. 4.4), while $\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$ remains still bounded and small, see Fig. 7b.

An attempt to calculate the hull of the PSS of Eq. (10) by calculating the hull of PVSS, that is by solving 2^k systems of real equations for all combinations of endpoints of the interval parameters p_i (i.e. for all members of vert $\mathbf{p}$), will not produce valid results unless it is assured that components of the solution vector $\mathbf{x}$ depend monotonically upon all p_i's. This is rarely the case: as is shown by the examples in Fig. 7, even when every coefficient of the system depends linearly and monotonically on a single interval parameter, the solution depends on the parameter $\mathbf{p}$ non-monotonically. Hence, in general, only the inclusion relation between hulls of PVSS and PSS holds, so that the hull of PVSS is an *inner* estimation of the hull of PSS,

$$\text{hull}\,\Sigma(\mathbf{A}(\text{vert}\,\mathbf{p}), \mathbf{b}(\text{vert}\,\mathbf{p})) \subseteq \text{hull}\,\Sigma(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p})) \subseteq \text{hull}\,\Sigma(\mathbf{A}, \mathbf{b}),$$

provided the appropriate hulls exist—e.g., hull $\Sigma(\mathbf{A}, \mathbf{b})$ does not exist for the example in Fig. 7b since $\Sigma(\mathbf{A}, \mathbf{b})$ in this case is unbounded (or, alternatively, we may consider it to be equal to the whole Ox_1x_2 plane—in this way the above inclusions are also valid for unbounded solution sets).

6.2. Linear and symmetric solution sets

There are only a few results concerning solving the general problem of finding interval estimates for functions, or linear interval systems of equations, depending on interval parameters. When the dependence of the coefficients of the system matrices $\mathbf{A}$ and $\mathbf{b}$ on interval parameters $\mathbf{p}$ is linear, i.e. when $a_{ij}(\mathbf{p})$ and $b_i(\mathbf{p})$ are linear functions of $\mathbf{p} \in \mathbb{R}^k$, the corresponding PSS is called *linear parametric solution set* (LPSS) and denoted as $\Sigma_{\text{lin}}(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$. General characterizations of shapes of the $\Sigma_{\text{lin}}(\mathbf{A}(\mathbf{p}), \mathbf{b}(\mathbf{p}))$ solution set were not investigated yet. Note that both examples in Fig. 7 are linear parametric. The first attempt to develop an algorithm for inner and outer estimation of the LPSS has been reported by Rump [22], as a generalization of Jansson's method for symmetric matrices [6].

An important special case of the linear dependence is given when the coefficients of the matrix $\mathbf{A}$ are considered as parameters (components of the vector $\mathbf{p}$), but the matrix is required to be always symmetric, i.e. $\mathbf{A}(\mathbf{p}) = \mathbf{A}(\mathbf{p})^{\mathrm{T}}$ and thus $a_{ij}(\mathbf{p}) = a_{ji}(\mathbf{p}) = p_{ij}$, $b_i(\mathbf{p}) = p_i$, $p = \{p_{ij}\} \cup \{p_i\}$. Such a situation occurs for many linear mechanical systems, e.g. trusses [14]. In this case, the solution set is defined as

$$\Sigma_{\text{sym}}(\mathbf{A}, \mathbf{b}) = \{\tilde{\mathbf{x}} \in \mathbb{R}^n \mid (\exists \tilde{\mathbf{A}} \in \mathbf{A},\, \tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{\mathrm{T}})(\exists \tilde{\mathbf{b}} \in \mathbf{b})\, \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}\}, \qquad \mathbf{A} = \mathbf{A}^{\mathrm{T}}. \tag{11}$$

Jansson [6] describes the method working with this symmetry constraint. Figure 8 shows an example of an interval system (due to Behnke, also used by Rump [22]), with the symmetric solution set marked. Note the curved boundaries of the set: according to [2], in general the symmetric solution set is bounded by quadratic curves or (hyper)surfaces.

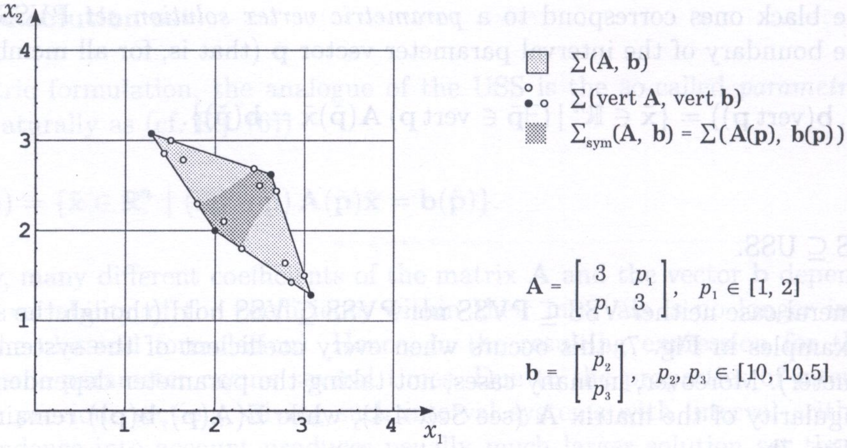

Fig. 8. A Rump-Behnke example [22] of a *symmetric solution set*

7. Conclusions

In the paper several basic types of solution sets for systems of interval equations have been presented and discussed. A special attention has been given to the construction and properties of the united solution set and the effects of coefficient dependence leading to the parametric solution set.

Systems of linear interval equations and their solutions constitute a much richer and more complex domain than classic systems of (real) linear equations. Hence, they require a much more complex analysis, including development of new analytical tools. Also, computational algorithms for finding the solutions or approximating them become more complex, which asks for searching for methods specialized to different types of problems. To achieve that goal, we need more comprehensive *characterization* and *classification* of various types of solutions.

The very important problem of *overestimation* of interval enclosures (or *coefficient dependence*) requires special treatment and construction of algorithms. Especially important, but still not solved satisfactorily, is here the problem of finding effective methods and algorithms for calculating the parametric solution sets and their approximations.

Another result of our work is the realization of the fact that in interval formulation the practical importance of *singular systems* is much greater than in applications of standard linear systems. Analysis and characterization of singular interval systems will require development of new formal and computational tools.

From the experience of the authors it also follows that *diagrammatic methods* of representation and analysis are very useful for more comprehensible characterization and understanding of the domain.

8. Acknowledgements

The work was supported by grant No. 8 T11F 006 15 (for years 1998-2001) from The State Committee of Scientific Research (KBN). Authors wish to thank the anonymous referee of this paper for insightful comments and advice.

References

[1] G. Alefeld, J. Herzberger. *Introduction to Interval Computations.* Academic Press, New York, 1983.
[2] G. Alefeld, V. Kreinovich, G. Mayer. The shape of the symmetric solution set. In: R.B. Kearfott, V. Kreinovich, eds., *Applications of Interval Computations.* Kluwer Academic Publishers, Dordrecht 1996.
[3] J.J. Buckley, Y. Qu. On using α-cuts to evaluate fuzzy equations. *Fuzzy Sets and Systems*, **38**: 309–312, 1990.
[4] E. Hansen. *Global Optimization Using Interval Analysis.* Marcel Dekker, New York, 1992.

[5] D.J. Hartfiel. Concerning the solution set of $Ax = b$ where $P \leq A \leq Q$ and $p \leq b \leq q$. *Numerische Mathematik*, **35**: 355–359, 1980.

[6] C. Jansson. Interval linear systems with symmetric matrices, skew-symmetric matrices and dependencies in the right hand side. *Computing*, **46**: 265–274, 1991.

[7] H.U. Köylüoğlu, A.Ş. Çakmak, S.R.K. Nielsen. Interval mapping in structural mechanics. In: Spanos, ed., *Computational Stochastic Mechanics*. 125–133, Balkema, Rotterdam 1995.

[8] V. Kreinovich, A. Lakeyev, J. Rohn, P. Kahl. *Computational Complexity and Feasibility of Data Processing and Interval Computations*. Kluwer Academic Publishers, Dordrecht, 1998.

[9] Z. Kulpa. Diagrammatic representation and reasoning. *Machine GRAPHICS & VISION*, **3**(1–2): 77–103, 1994.

[10] Z. Kulpa. Diagrammatic representation for a space of intervals. *Machine GRAPHICS & VISION*, **6**(1): 5–24, 1997.

[11] Z. Kulpa. Diagrammatic representation of interval space in proving theorems about interval relations. *Reliable Computing*, **3**(3): 209–217, 1997.

[12] Z. Kulpa. Diagrammatic representation of interval space. Part I: Basics; Part II: Arithmetic. Internal Report DIAR-1/99 and DIAR-2/99, IFTR PAS, Warsaw, 1999.

[13] Z. Kulpa. Diagrammatic representation for interval arithmetic. *Linear Algebra Appl.*, 2001 (to appear).

[14] Z. Kulpa, A. Pownuk, I. Skalna. Analysis of linear mechanical structures with uncertainties by means of interval methods. *CAMES*, **5**(4): 443–477, 1998.

[15] Z. Kulpa, A. Radomski, O. Gajl, M. Kleiber, I. Skalna. Hybrid expert system for qualitative and quantitative analysis of truss structures. *Engin. Appl. Artif. Intell.*, **12**: 229–240, 1999.

[16] R.E. Moore. *Interval Analysis*. Prentice Hall, Englewood Cliffs, NJ, 1966.

[17] A. Neumaier. *Interval methods for systems of equations*. Cambridge University Press, Cambridge, 1990.

[18] W. Oettli, W. Prager. Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides. *Numer. Math.*, **6**: 405–409, 1964.

[19] S.S. Rao, L. Berke. Analysis of uncertain structural systems using interval analysis. *AIAA Journal*, **35**(4): 727–735, 1997.

[20] J. Rohn. Systems of linear interval equations. *Linear Algebra Appl.*, **126**: 39–78, 1989.

[21] J. Rohn. Cheap and tight bounds: The recent result by E. Hansen can be made more efficient. *Interval Computations*, **4**: 13–21, 1993.

[22] S.M. Rump. Verification methods for dense and sparse systems of equations. In: J. Herzberger, ed., *Topics in Validated Computations*. 63–135, Elsevier Science B.V., 1994.

[23] S. P. Shary. Algebraic approach to the interval linear static identification, tolerance, and control problems, or one more application of Kaucher arithmetic. *Reliable Computing*, **2**(1): 3–33, 1996.

[24] Yu.I. Shokin. On interval problems, interval algorithms and their computational complexity. In: G. Alefeld, A. Frommer, B. Lang, eds., *Scientific Computing and Validated Numerics*. 314–328, Akademie-Verlag, Berlin 1996.

[25] J. Skrzypczyk. Fuzzy finite element methods—A new methodology. In: *Computer Methods in Mechanics (Proc. XIII Polish Conference on Computer Methods in Mechanics, Poznań, Poland, May 5-8, 1997)*, **4**: 1187–1194. Poznań University of Technology, Poznań 1997.

[26] M. Warmus. Calculus of approximations. *Bull. Acad. Pol. Sci., Cl. III*, **4**(5): 253–259, 1956.