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Generation of hybrid grids over plane domains

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The paper presents an algorithm of coverage 2-D multiconnected domain by trianlges and quadrilaterals. The zone covered with quadrilaterals is structured, thus the zone covered with triangles is unstructured. The density of the structured grid is controlled only on one of the surrounding loop, on which the points are generated with mesh density function. Unstructured zone is triangulated by using Delaunay advancing front technique triangulation of points previously generated.

1. Introduction

The main task of the paper is to work out an algorithm and appropriate computer code for generation of quadrilateral and triangular meshes covering a plane domain. Additionally, it is assumed, that quadrilateral meshes will be structured and triangular unstructured [1]. This assumption gives the possibility to create so called hybrid [9, 11] mesh for which some parts of the domain are covered with triangles and some parts of the domain with quadrilaterals. The both kinds of meshes should have some common edges. It requires to prepare in some specific way appropriate data structure.

Hybrid grids have applications in variety of fields of boundary value problem [12], especially in fluid mechanics where for some zones (boundary layers) surrounding airfoil profiles it is recommended the generation of quadrilateral structured meshes, whereas triangular unstructured [7] mesh fills the remaining part of the domain [4].

Structured mesh fills the part of the domain contained between two closed non-intersecting and without multiples points curves. The size the of the mesh is controlled by a given mesh density function defined in the closure of the domain, but in case of structured quadrilateral grid is restricted only to one of the surrounding loops, it follows from the nature of structured mesh.

The implementation of hybrid is the subject of research of last years [11]. The solvers for finite element method based upon structured grid are more effiecient than those on unstructured grid [10]. High aspect ratio cells necessary for the resolution of viscous boundary layers are necessary around the profile. In [10] the structure around the profile and ustructured in the remainder of the domain is preferred too, but points of the structured grid are generated over normals to the internal curve. For hybrid generation the multi-block technique is used [11], in some blocks structured grid is generated in some unstructured.

The proposed algorithm is connected with previously evaluated unstructured grid generator of triangular meshes [6] where advancing front technique is combined with Delaunay triangulation [1, 7].

The whole algorithm can be divided into the following stages:

- points generation over curves with given density,
- algorithmic construction of the mapping from unit $[0,1] \times [0,1]$ square onto subdomain contained between two non-intersecting, without multiple points curves,
- generation of unstructured grid in the unstructured part of the subdomain by advancing front technique,

- Delaunay triangulation of obtained unstructured grid,
- combining both structured and unstructured meshes by taking appropriate number of nodes for common part of both grids.

2. Representation of the 2-D domain

It is assumed, that domain, which is triangulated, can be multiconnected, with arbitrary, but finite number of internal holes. Every connected component of the boundary is called a loop. The topoplogy of the domain and what follows, appropriate data will be connected with those loops. There is exactly one external loop and all the others are internal loops creating holes (see Fig. 1). Every loop consists of some primitive curves.

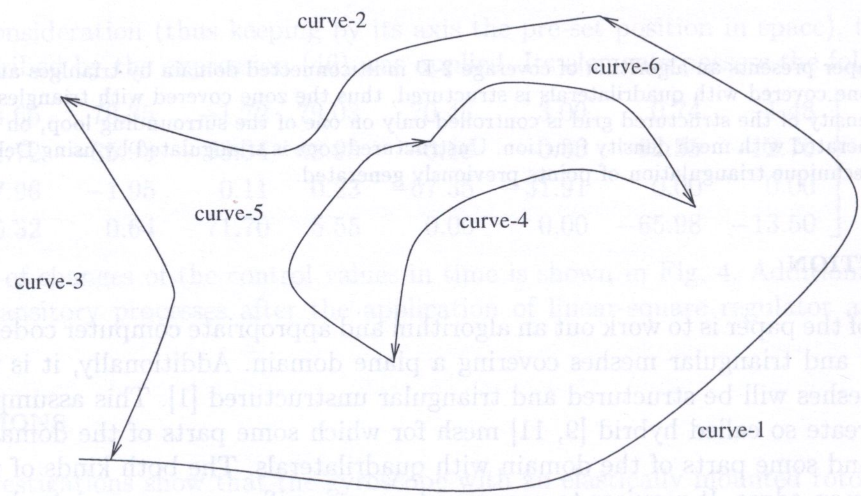

Fig. 1. Boundary representation

All the loops of the boundary are represented by a collection of curves, every curve is topologically equivalent to straight line segment and is defined by its ends and type.

The following types of curves can create the boundary of the domain:

- straight-line segment,
- arch of circle,
- B-spline curve.

Straight-line segment is represented by its ends only. Additionally arch of circle is represented by its ends and coordinates of the centre, thus B-spline curve is represented as a sequence of coordinates of vertices, which are leading points of the curve. Every loop consits of curves in the order that the end of one is simultaneously the beginning of the next. The curves are ordered in counter-clock wise fashion with respect to the domain.

The topology of the considered domain is represented in the hierarchical order with respect to the size of 2-D sets. It means, that curves are defined by points (points are primitives), loops are defined by curves, and domain is defined by loops. So the hierarchy is as follows:

- point,
- curve,
- loop,
- boundary and domain.

3. The overall algorithm of hybrid grid generation

Structured grids are performed by effective construction of mapping of a mesh spread over primitive domain (usually *square, triangle*) onto considered domain. This approach restricts the class of the available domains, but gives meshes on which some operations like finding neighbors are simplified and structured grids give possibility to make more effective solver.

For the extension the class of domains on which structured grids are generated, it is proposed to fill with quadrilaterals domain contained between two loops i.e. the domain topologically equivalent to the ring, additionally, every of these loops may contain B-spline curve in its description.

The presented algorithm of hybrid grid generation has the following stages:

- generation of points with the given mesh density function over all curves consisting the boudary,
- generation of structured grids for specified zones,
- internal points generation in all the unstructured zones by advancing front technique with the given mesh density function, provided, that as the initial front boundary segments and segments on the external loops of unstructured zones are taken,
- Delaunay advancing front triangulation of unstructured part of the domain,
- Laplacian smoothing of the unstructured mesh,
- preparing output data structure combining both kinds of the meshes.

3.1. The construction of the mapping from unit square $[0,1] \times [0,1]$ onto domain contained between two loops

The main idea of the structured grid generation is based upon construction of the mapping from unit square onto considered domain, and then by using this mapping, transfering the mesh from the square onto the domain. The main points of structured grid generation are:

- points generation on all curves appearing in the description of structured part of the considered domain,
- taking for the internal (or alternatively external) curve of the boundary of structured grid obtained points as vertices of structured grid,
- the curve $r_1(t)$ description as close broken line passing through successively previously generated boundary points,
- the curve $r_2(t)$ description as B-spline curve defined by obtained boundary points,
- normalization of the parameters of curves $r_1(t)$ and $r_2(t)$ to the straight line segment $[0,1]$ as reference domain,
- the construction of the mapping from $[0,1] \times [0,1]$ onto considered domain.

The curve $r(t)$ is described as follows. Provided that set of points $\{P_i\}_{i=0}^{N_1}$ is the set of points generated on the first curve, and defining

$$m = \sum_{j=0}^{N_1-1} |P_j P_{j+1}|, \tag{1}$$

and

$$t_i = \sum_{j=0}^{i-1} |P_j P_{j+1}| , \tag{2}$$

then

$$r(t) = \frac{t_{i-1} - t}{t_{i-1} - t_i} P_i + \frac{t - t_i}{t_{i-1} - t_i} P_{i-1} , \tag{3}$$

if $t \in [t_{i-1}, t_i]$, for $i = 1, \ldots, N_1 - 1$. Because of Eqs. (1), all the intervals $t \in [t_{i-1}, t_i]$ fill the whole interval $[0, m]$ so the parameter t can vary in it.

Reparametrized curve $r'(t)$ is then defined,

$$r'(t) = r'(tm), \tag{4}$$

where $0 \leq t \leq 1$. Obviously

$$r'(\frac{t_i}{m}) = P_i, \tag{5}$$

for $i = 0, 1, \ldots, N_1$.

On the other hand, if Q_i for $i = 0, 1, \ldots, N_2$, where $Q_0 = Q_{N_2}$, are the points generated over the second curve then the B-spline curve is defined taking these points as the nodes, we define appropriately curves $s'(t)$ and $s(t)$ by the following formula [2],

$$s(t) = \sum_{i=1}^{N_e} Q_i B_{i,p}(t), \tag{6}$$

with knot vector

$$V = \{\tau_0 , \ldots , \tau_0 , \tau_1 , \ldots , \tau_{m-1} , \tau_m , \ldots , \tau_m\} , \tag{7}$$

where $\tau_0 = 0$, and

$$\tau_i = \sum_{j=0}^{i-1} |Q_i Q_{i+1}| \tag{8}$$

when $i = 1, \ldots, N_2$.

Thus, B-spline functions [2] $B_{i,p}$ of order p are defined by the recursive formula as follows,

$$B_{i,p}(t) = \frac{(t - \tau_i) B_{i,p-1}(t)}{\tau_{i+p-1} - \tau_i} + \frac{(\tau_{i+k} - t) B_{i+1,p}(t)}{\tau_{i+p} - \tau_{i+1}} , \tag{9}$$

for $p > 0$, and

$$B_{i,0}(t) = \begin{cases} 1 & \text{if } \tau_i \leq t < \tau_{i+1} , \\ 0 & \text{otherwise,} \end{cases}$$

if $p = 0$.

By analogy to $r(t)$,

$$s'(t) = s(tM), \tag{10}$$

where $0 \leq t \leq 1$, thus

$$M = \sum_{j=0}^{N_2} |Q_j Q_{j+1}| \tag{11}$$

is a set of generated points on the internal loop, and we have

$$s'(\tau_i) = Q_i \,. \tag{12}$$

Now a mapping can be defined from unit square into $\mathbb{R}^2$,

$$H : [0,1] \times [0,1] \mapsto \mathbb{R}^2 , \tag{13}$$

as

$$H(u,v) = r'(u)(1-v) + s'(u)v. \tag{14}$$

It is easy to check that

$$H(u,0) = r'(u), \tag{15}$$

$$H(u,1) = s'(u). \tag{16}$$

The vertices of the stuctured quadrilateral mesh are defined as follows,

$$P_{ij} = H\left(\frac{t_i}{N_1}, \frac{j}{k}\right). \tag{17}$$

for $i = 0, 1, \ldots, N_1$ and $j = 0, 1, \ldots, k$, with k equal to the number of layers of structured grid.

3.2. The algorithm of unstructured grid generation

The algorithms of unstructured grids generations were evaluated by the author of the paper for the last years [4, 6]. The advancing front technique and Delaunay advancing front method [8] were applied for points generation and triangulation.

After generation of points Delaunay advancing front technique is used to triangulate the domain [9]. In case of convex domains the Delaunay advancing front technique leads to Delaunay triangulation [3, 6], but it can be proved, that this approach is a right generalisation of Delaunay triangulation onto non-convex and multiconnected domains.

The algorithm of unstructured grid generation consists of the following steps:

- taking obtained points on curves as nodes of unstructured grid and associated linear segments as starting front combined together with previously generated points and appropriate linear segments of curves being the boundary of unstructured part of the domain,
- Generation of internal points with given density by advancing front technique,
- Triangulation of the obtained set of points by Delaunay advancing front technique,
- Laplacian smoothing of the mesh.

4. Numerical results

The appropriate program was made, which realizes the algorithm proposed in the paper. The program was combined with evaluated for last years program [4, 6] of unstructured grids generation. The common nodes for structured and unstructured grid generation have that same numbers. Thanks to that the whole mesh is connected, and can be directly used for finite element applications. Many examples were performed to test the made code. It is especially important to grid domains having B-spline curves in their boundary definition, so all the presented numerical examples satisfy the conditions.

In Fig. 2 pure structured grid is presented wrapped around airfoil NACA0012. In Fig. 3 airfoil profile BINACA0012 is covered by the hybrid grid, where both profiles have 2 layers of structured grid, the mesh density function was taken as an affine function of distance from heads and tails of the profiles when current point of the domains is close to them.

In Fig. 4 hybrid grid over NACA0012 with structured around profile is presented. Thus, Figs. 5, 6 present gridded BINACA0012 profiles. In the first case we have two layers of quadrilateral mesh over one profile and five over the other. In second case there are two layered structured grids surounding both the profiles.

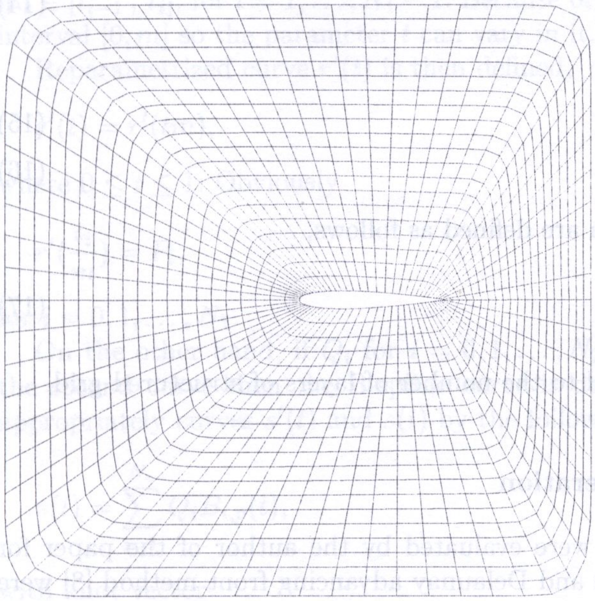

Fig. 2. Pure structure grid over airfoil profile NACA0012

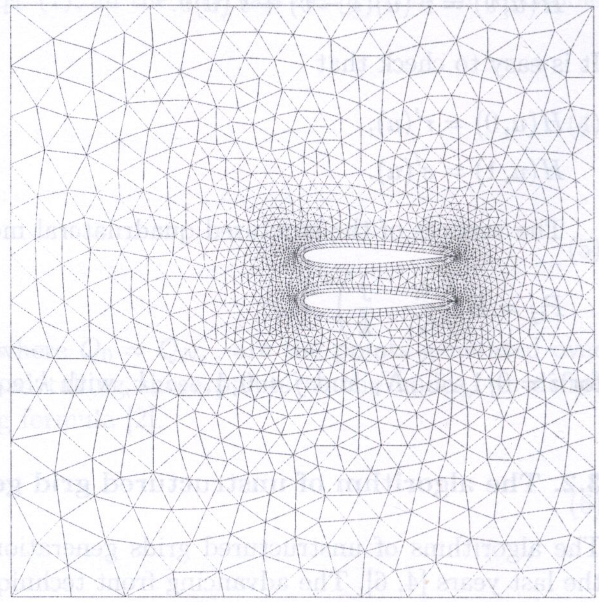

Fig. 3. Hybrid mesh over BINACA0012 with two layers of structured grid over both profiles

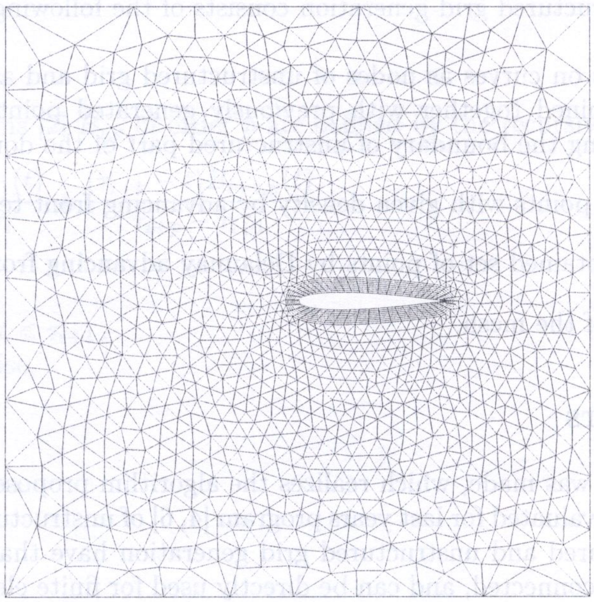

Fig. 4. Hybrid grid NACA0012 airfoil with 8 layers of quadrilateral structured grid over profiles

In Figs. 7, 8 there are presented hybrid meshes with different structured zones then in previous cases. Different mesh density functions were used.

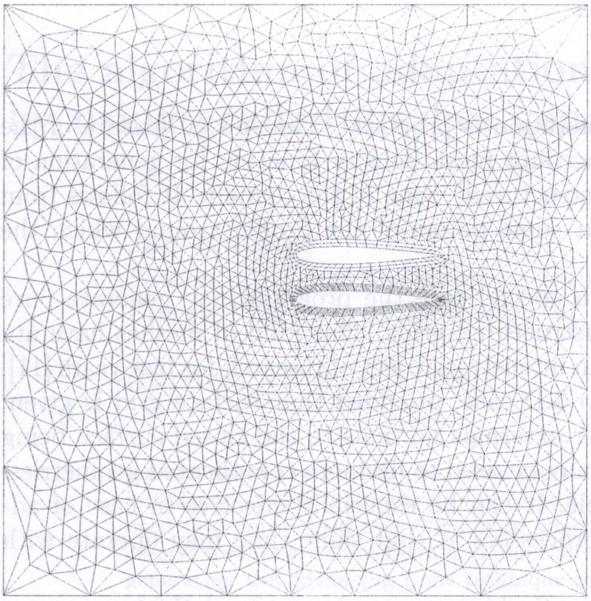

Fig. 5. Hybrid grid over BINACA0012 airfoil with appropriately 5 and 2 layers of quadrilateral structured grid over profiles

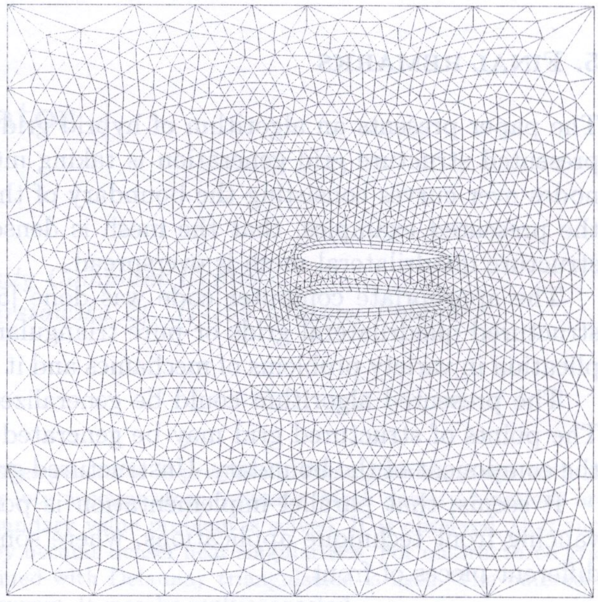

Fig. 6. Hybrid grid over BINACA0012 airfoil with 2 layers of quadrilateral structured grid over both profiles

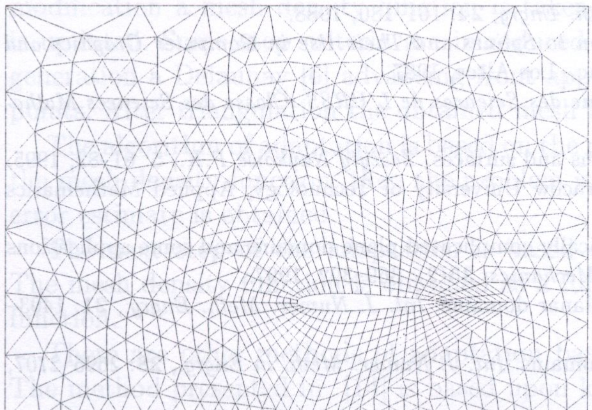

Fig. 7. Hybrid grid over NACA0012 airfoil with another external loop of structured zone

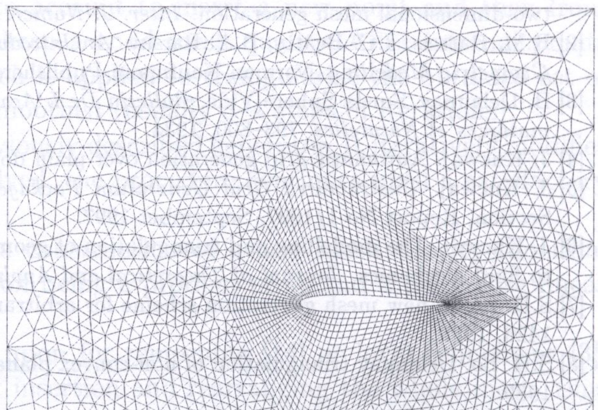

Fig. 8. Hybrid grid over NACA0012 airfoil with another mesh density function

In Figs. 7, 8 there are presented hybrid meshes with different structured zones than in previous cases. Different mesh density functions were used.

5. Final remarks

The paper presents an algorithm of hybrid grid generation is proposed. The special feature of the presented structure grid generation is taking into account mesh density function for generation of nodes over curves consisting the boundary of the "structured" zone of the domain. It follows from the presented examples, that the mesh is denser at some points of the profile around which the structure quadrilateral mesh is overspread.

The appropriate computer code was done [5]. In special case the program may generate pure structured quadrilateral grids or pure triangular unstructured grids. This kind of meshes have application in variety problems of mechanics and it is an intensive subject of research for last years [11].

The presented algorithm could be generalized to 3-D case, provided that surface grid is given. The points of structured grid may be generated over externals normals to the closed surface being the component of the boundary. Unstructured grid may be used to fill more complicated geometries.

It seems, that parallel implementation is possible. It can be especially effective in case of advancing front technique applications, where some processors at that same time may operate with different pieces of front.

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