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Modelling of heat transfer in biological tissue by interval FEM

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In this paper, an algorithm of calculation of extreme values of temperature based on interval arithmetic is presented. Many mechanical systems with uncertain parameters $\lambda \in \Lambda$ can be described by a parameter dependent system of linear equations $K(\lambda)T = B(\lambda)$. Using natural interval extension of a real function, one can transform the system of linear equations into the system of linear interval equations $\mathbf{K}(\mathbf{\Lambda})\mathbf{T} = \mathbf{B}(\mathbf{\Lambda})$. Solution of the system of linear interval equations always contains the exact solution of the parameter dependent system of equations. A new method of computation of extreme values of mechanical quantities based on the monotonicity test is introduced. This method can give exact solution of a parameter dependent system of equations.

1. Introduction

The study of the influence of parameters upon behaviour of mathematical models is one of the basic problems of computational mechanics. Usually, one is interested in systems which are locally stable in the sense that their qualitative behaviour does not change under small variations of the parameters. Here, some form of perturbation theory may be the appropriate tool [29]. One often needs to know explicitly the properties of the solutions for a large region of physical significance. The mathematical models for describing the systems considered in this paper are in the form [23, 29]

$$F(z,t) = 0 \tag{1}$$

where $F : Z \times T \rightarrow Z$. Space Z characterizes the state of the system, t denotes the parameter variable allowed to vary over the space T.

Consider the mechanical system (1) with uncertain parameters t. If sufficient experimental data are available, probabilistic methods can be applied [33, 34]. Alternatively, the convex model of uncertainty can be applied [2–4, 6, 36].

One of the simplest ways of representation of uncertain or inexact data, as well as of inexact computations with them, is based on interval arithmetic [1, 18, 20, 23, 35]. Other methods are based on set valued analysis [31] and classical theory of optimization [2, 3, 5, 7, 9, 19, 34, 37]. Convex model of uncertainty can be represented also by ellipses of uncertainty [8, 26, 36].

When function is sufficiently smooth, to calculate its extreme values, the Kuhn–Tucker condition can be used. When function $z(t)$ is given explicitly, one can use interval global optimization [32] or other global optimization method.

2. Interval Arithmetic

A real interval is a set of real numbers such that

$$\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in R : \ \underline{x} \le x \le \overline{x}\}. \tag{2}$$

The set of all intervals is denoted by IR [1, 23] and called a real interval space. Operations and functions on reals are naturally extended to interval operands according to the general formulas [20]

$$\mathbf{x} \oplus \mathbf{y} = \{x \oplus y : \quad \mathbf{x} \in x, \ \ y \in \mathbf{y}\} \qquad \text{where} \qquad \oplus \in \{+, -, \cdot, /\} \tag{3}$$

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \{f(x_1, \ldots, x_n) : \quad x_1 \in \mathbf{x}_1, \ \ldots, \ x_n \in \mathbf{x}_n\}. \tag{4}$$

The function f is called *programmable* if $f(x)$ can be built up from arithmetic, logical and comparison operators and some collection of standard transcendental functions (like sin, cos, power, etc.). Given an argument x, the function value $f(x)$ can be computed with a finite number of operations [23]. All the functions in this paper are assumed to be programmable.

Another important property of arithmetic operations on intervals is called *inclusion isotonicity*

$$(\mathbf{a} \subseteq \mathbf{c}) \wedge (\mathbf{b} \subseteq \mathbf{d}) \Rightarrow \mathbf{a} \oplus \mathbf{b} \subseteq \mathbf{c} \oplus \mathbf{d} \tag{5}$$

that is, the result of straightforward calculation of interval expression will always include the proper result ($\oplus$ is any interval arithmetic operation).

Let $\mathbf{x} \in IR$, then the natural interval extension $\boldsymbol{f}$ of a programmable function f to $\mathbf{x}$ is defined as an expression which is obtained from the expression $f(x)$ by replacing each occurrence of the variable x by $\mathbf{x}$, the arithmetic operations of R by the corresponding interval arithmetic operations, and each occurrence of a pre-declared function g by the corresponding inclusion function $\boldsymbol{g}$:

$$\{g(x) : \ x \in \mathbf{x}\} \subseteq \{\boldsymbol{g}(x) \ \ x \in \mathbf{x}\}. \tag{6}$$

Every inclusion function $\boldsymbol{f}(\mathbf{x})$ has the property:

$$x \in \mathbf{x} \qquad \text{implies} \qquad f(x) \in \boldsymbol{f}(\mathbf{x}). \tag{7}$$

Property (7) is the key to almost all interval arithmetic applications and results [1] and should be called the *fundamental property of interval arithmetic.*

For any bounded set of real numbers S, one can define a *smallest interval enclosure* of the set

$$\text{hull}\, S = [\inf S, \sup S]. \tag{8}$$

In the same way one can define the space of multidimensional intervals IR^n

$$\mathbf{x} \in IR^n \Leftrightarrow \mathbf{x} = \mathbf{x}_1 \times \mathbf{x}_2 \times \ \times \mathbf{x}_n \qquad \text{where} \qquad \mathbf{x}_i \in IR. \tag{9}$$

3. Systems of Linear Interval Equations

Let us consider a linear interval system of equations with an interval coefficient matrix $A \in IR^{n \times n}$ and an interval right-hand vector $B \in IR^n$ [23]:

$$\mathbf{AX} = \mathbf{B}. \tag{10}$$

The solution set of Eq. (10) is defined as:

$$\sum(\mathbf{A}, \mathbf{B}) = \{X \in R^n : \quad \exists A \in \mathbf{A}, \ \ \exists B \in \mathbf{B}, \ \ A \cdot X = B\}. \tag{11}$$

Calculating (and representing) the solutions set $\sum(\mathbf{A}, \mathbf{B})$ may be quite hard and impractical, especially for larger n. Therefore, for many practical purposes, one is satisfied with the *interval enclosure* of the solution set (11). The smallest enclosure is the hull of the set

$$\text{hull} \sum(\mathbf{A}, \mathbf{B}) = \left[\inf \sum(\mathbf{A}, \mathbf{B}), \sup \sum(\mathbf{A}, \mathbf{B})\right]. \tag{12}$$

There are many methods for solving Eq. (10). The simplest method is the use of all combinations of the endpoints of intervals of the matrix $\mathbf{A}$ and vector $\mathbf{B}$ [18]. Others, like Rohn sign-accord algorithm [30] or linear programming method [15, 18] are based on the theorem of Oettli and Prager [23]. Some methods give only an interval estimation of the set [18, 23]. Computation of the exact solution set or its hull ((11) or (12)) is NP-hard [17].

4. Interval FEM

In this paper extreme values of temperature in biological tissue using Interval Finite Element Method [15, 16, 21, 22, 27, 28] are calculated. The Pennes equation describing the steady temperature field in a biological tissue is considered [10]:

$$\operatorname{div}\left[\lambda \operatorname{grad} T(X)\right] + Q_{\mathrm{met}} + Q_{\mathrm{perf}} = 0 \tag{13}$$

where λ is the thermal conductivity of tissue, Q_{met} is the metabolic heat source, Q_{perf} is the perfusion heat source, T is the temperature. Equation (13) is supplemented by boundary conditions which can be written in the form

$$X \in \Gamma : \quad \Phi\left[T(X), \frac{\partial T(X)}{\partial n}\right] = 0. \tag{14}$$

The perfusion heat source is as follows

$$Q_{\mathrm{perf}}(X) = c_b \rho_b G_b [T(X) - T_b] \tag{15}$$

where c_b, ρ_b are the specific heat and mass density of blood, T_b is the temperature of blood and G_b is the perfusion rate. In this paper the one-dimensional problem is considered. In order to obtain the solution, the FEM is applied. Finite Element Method leads to the following system of equations

$$K(\lambda)T = B(\lambda) \tag{16}$$

where K is the global heat conductivity matrix, T is the vector of unknown temperatures and B is the vector containing the information about boundary conditions. The exact solution set of (16) can be described as

$$\{(T_0(\lambda), \dots, T_n(\lambda)) : \ \lambda \in \mathbf{\Lambda}\} = \{(T(x_0, \lambda), \dots, T(x_n, \lambda)) : \ \lambda \in \mathbf{\Lambda}\} \qquad \text{where} \quad \mathbf{\Lambda} = [\underline{\lambda}, \overline{\lambda}] . \tag{17}$$

Interval analysis provides a rigorous and realistic sensitivity analysis of the solution of (17) under perturbations of arbitrary specified magnitude (i.e. not only asymptotically for "sufficiently small" perturbations). Computation of the exact solution set (17) is very difficult [11, 14, 23–25]. Uncertainty is introduced by assigning an interval valued parameters using their interval extensions [1]. Then the system of equations (16) becomes a system of interval equations in the form

$$\mathbf{K}(\mathbf{\Lambda})\mathbf{T} = \mathbf{B}(\mathbf{\Lambda}). \tag{18}$$

From the fundamental property of interval arithmetic [23], it follows that [12, 23]

$$\{(T_0(\lambda), \dots, T_n(\lambda)) : \ \lambda \in \Lambda\} = \{(T(x_0, \lambda), \dots, T(x_n, \lambda)) : \ \lambda \in \Lambda\} \subseteq \sum (\mathbf{K}(\mathbf{\Lambda}), \mathbf{B}(\mathbf{\Lambda})) \tag{19}$$

i.e. the solution of the system of linear interval equations (18) always contains the exact solution set (17) in a nodal point x_i $(i = 0, \dots, n)$. Both solution sets (17), (19) are usually very complicated. Because of this, in applications, one uses the smallest interval which contains the exact solution set ((17) or (19)) [18].

$$\operatorname{hull} \sum (\mathbf{K}(\mathbf{\Lambda}), \mathbf{B}(\mathbf{\Lambda})) \in IR^n. \tag{20}$$

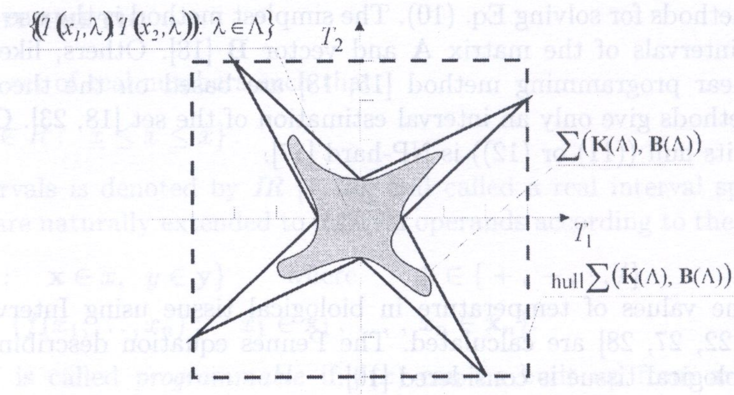

Fig. 1. Solutions sets of equations (17) and (19)

5. Interval FEM based on monotonicity test

In many engineering problems, the solution $T_i(\lambda)$ is monotone in $\mathbf{\Lambda}$. In this case extreme values of mechanical quantities can be calculated using endpoints of the interval $\mathbf{\Lambda}$

$$\underline{T}_i = \min\left\{T_i\left(\underline{\lambda}\right), T_i\left(\overline{\lambda}\right)\right\}, \qquad \overline{T}_i = \max\left\{T_i\left(\underline{\lambda}\right), T_i\left(\overline{\lambda}\right)\right\} \tag{21}$$

From the fundamental property of interval arithmetic (7) and properties of he derivatives, it follows that

$$\text{if} \quad 0 \notin \frac{\partial \mathbf{T}_i(\mathbf{\Lambda})}{\partial \lambda} \quad \text{then} \quad T_i(\lambda) \ \text{ is monotone in } \mathbf{\Lambda}. \tag{22}$$

The derivative of temperature T_i can be calculated using the implicit function theorem

$$K\frac{\partial T}{\partial \lambda} = \frac{\partial B}{\partial \lambda} - \frac{\partial K}{\partial \lambda}T. \tag{23}$$

An interval extension of derivative (22) can be calculated as a solution of the following system of linear interval equations

$$\mathbf{K}(\mathbf{\Lambda})\frac{\partial \mathbf{T}}{\partial \lambda} = \frac{\partial \mathbf{B}(\mathbf{\Lambda})}{\partial \lambda} - \frac{\partial \mathbf{K}(\mathbf{\Lambda})}{\partial \lambda}\mathbf{T}(\mathbf{\Lambda}) \tag{24}$$

where

$$\mathbf{T}(\mathbf{\Lambda}) = \text{hull} \sum (\mathbf{K}(\mathbf{\Lambda}), \mathbf{B}(\mathbf{\Lambda})). \tag{25}$$

From the fundamental property of interval arithmetic (7), Eqs. (24) and (22), it follows that if

$$0 \notin \frac{\partial \mathbf{T}(\mathbf{\Lambda})}{\partial \lambda} = \sum\left(\mathbf{K}(\mathbf{\Lambda}), \frac{\partial \mathbf{B}(\mathbf{\Lambda})}{\partial \lambda} - \frac{\partial \mathbf{K}(\mathbf{\Lambda})}{\partial \lambda}\mathbf{T}(\mathbf{\Lambda})\right) \tag{26}$$

then the functions $T_i(\lambda)$ are monotone in $\mathbf{\Lambda}$. If the interval $\mathbf{\Lambda}$ is too large, one can divide it into parts Λ_i such that $\mathbf{\Lambda} = \bigcup_i \Lambda_i$ and $\text{int}(\Lambda_i) \cap \text{int}(\Lambda_j) = \emptyset$ for $i \neq j$. If the functions $T_i(\lambda)$ are monotone in all parts Λ_i then they are monotone in $\mathbf{\Lambda}$.

6. Mathematical description of the process and computations

A temperature field in biological tissue in cylindrical co-ordinate satisfies the following equations

$$\begin{cases} R_1 < r < R_2 : \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\lambda\dfrac{\mathrm{d}T(r)}{\mathrm{d}r}\right) + Q = 0, \\ r = R_1 \qquad : -\lambda\dfrac{\mathrm{d}T(r)}{\mathrm{d}r} = \alpha(T(r) - T_b), \\ r = R_2 \qquad : T(r) = T_t\,, \end{cases} \tag{27}$$

R_1, R_2 – internal and external radii of the domain,

α – heat transfer coefficient,

T_t – tissue temperature,

$Q = Q_{\text{met}} + Q_{\text{perf}}$ – constant value.

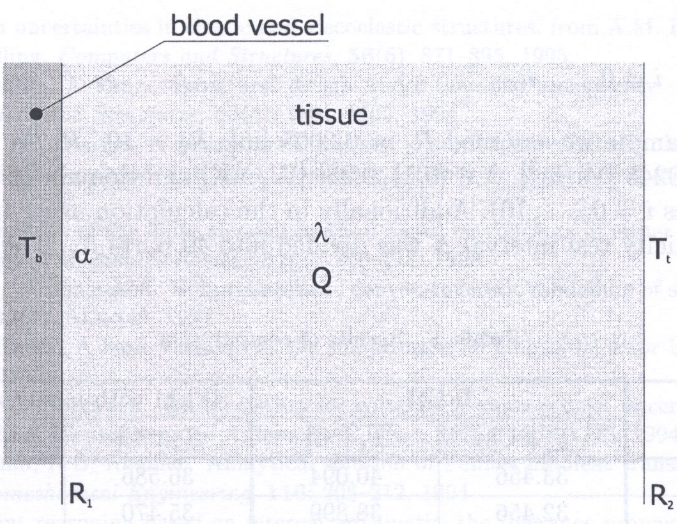

Fig. 2. The domain considered

Using the weighted residual method criterion one has

$$\int_{R_1}^{R_2}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r\lambda\frac{\mathrm{d}T}{\mathrm{d}r}\right) + rQ\right] w(r)\,\mathrm{d}r = 0. \tag{28}$$

and after integrating the first component of equation above by parts, one obtains the following equation

$$r\lambda\frac{\mathrm{d}T}{\mathrm{d}r}\,w(r)\bigg|_{R_1}^{R_2} - \int_{R_1}^{R_2} r\lambda\frac{\mathrm{d}T}{\mathrm{d}r}\frac{\mathrm{d}w(r)}{\mathrm{d}r}\,\mathrm{d}r + \int_{R_1}^{R_2} rQw(r)\,\mathrm{d}r = 0. \tag{29}$$

The integrals from R_1 to R_2 are substituted by the sum of integrals

$$\left[r\lambda\frac{\mathrm{d}T}{\mathrm{d}r}\,w(r)\right]_{R_2} - \left[r\lambda\frac{\mathrm{d}T}{\mathrm{d}r}\,w(r)\right]_{R_1} - \sum_{\substack{(e)\\ i=1}}^{n}\int_{r_{i-1}}^{r_i} r\lambda\frac{\mathrm{d}T}{\mathrm{d}r}\frac{\mathrm{d}w(r)}{\mathrm{d}r}\,\mathrm{d}r + \sum_{\substack{(e)\\ i=1}}^{n}\int_{r_{i-1}}^{r_i} r\,Q\,w(r)\,\mathrm{d}r = 0. \tag{30}$$

Temperature at the finite element sub-domain is described by the linear function

$$r \in [r_{i-1}\ \ r_i] : \ T(r) = \frac{r_i - r}{h}T_{i-1} + \frac{r - r_{i-1}}{h}T_i = N_{i-1}T_{i-1} + N_iT_i\,. \tag{31}$$

From Eq. (30), one obtains a system of linear equations in the form

$$K(\lambda)T = B(\lambda). \tag{32}$$

The coefficients of the global heat conductivity matrix for $i = 1, \ldots, n-1$ are

$$k_{0,0} = r_1^2 - r_0^2, \qquad k_{0,1} = r_0^2 - r_1^2, \tag{33}$$

$$k_{i,i-1} = r_{i-1}^2 - r_i^2, \qquad k_{i,i} = r_{i+1}^2 - r_{i-1}^2, \qquad k_{i,i+1} = r_i^2 - r_{i+1}^2, \tag{34}$$

$$k_{n,n-1} = r_{n-1}^2 - r_n^2, \qquad k_{n,n} = r_n^2 - r_{n-1}^2. \tag{35}$$

The coefficients of the vector B are

$$p_i = 4r_i^3 + r_{i-1}^3 + r_{i+1}^3 - 3r_i^2(r_{i-1} - r_{i+1}) \quad \text{for} \quad i = 1, \ldots, n-1, \tag{36}$$

$$p_0 = r_1^3 + r_0^2(2r_0 - 3r_1) + \frac{3R_1 \alpha T_b \lambda}{Qh}, \tag{37}$$

$$p_n = \frac{3T_t\lambda}{Qh}, \tag{38}$$

Finally

$$b_i = \frac{Qh}{3\lambda} p_i \quad \text{for} \quad i = 0, \ldots, n. \tag{39}$$

For a numerical example we assumed $R_1 = 0.0005$ [m], $R_2 = 10 \cdot R_1$, $\alpha = 2000$, $T_b = 37$ [°C], $T_t = 32.5$ [°C], $Q = 10245$ [W/m^3], $\lambda \in [0.21, 0.23]$ [W/mK] and domain of the tissue was divided into 10 elements (nodes $i = 0, \ldots, 10$). Additionally in the calculation using Interval Finite Element Method with monotonicity test interval $\mathbf{\Lambda}$ was divided into 40 parts Λ_i. The results of calculations are shown in Table 1.

Table 1. Results of computation

Number of node	IFEM		IFEM with monotonicity test	
	$\underline{T}_i$ [°C]	$\overline{T}_i$ [°C]	$\underline{T}_i$ [°C]	$\overline{T}_i$ [°C]
0	33.456	40.094	36.586	36.619
1	32.456	38.899	35.470	35.494
2	31.849	38.172	34.782	34.800
3	31.401	37.638	34.284	34.298
4	31.059	37.231	33.894	33.905
5	30.770	36.887	33.573	33.582
6	30.516	36.586	33.302	33.308
7	30.287	36.316	33.065	33.070
8	30.068	36.057	32.857	32.859
9	29.863	35.816	32.669	32.671
10	29.668	35.588	32.500	32.500

7. Conclusions

The Interval Finite Element Method (IFEM) always gives solutions in the form [15, 18]

$$\mathbf{T}(\mathbf{\Lambda}) = \text{hull} \sum (\mathbf{K}(\mathbf{\Lambda}), \mathbf{B}(\mathbf{\Lambda})). \tag{40}$$

This interval always contains the exact solution set of the parameter dependent system of equations (16). Both solutions are very similar only if the width of the interval $\mathbf{\Lambda}$ is very small. In other cases, the so called overestimation problem [13, 18] causes the Interval FEM to give very overestimated results. Coefficient dependence is the main source of overestimation [18]. This error is an integral part of IFEM and it is impossible to avoid this effect in the algorithm presented in Section 4. IFEM with monotonicity test gives the exact solution of Eq. (16). One can apply this algorithm in situations where parameter dependent solution is monotone. In other case it can be difficult to obtain the solution using this method.

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