Computer Assisted Mechanics and Engineering Sciences, 7: 471–478, 2000.

Finite element method for a nonlinear problem

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(Received November 3, 1999)

We consider the nonlinear eigenvalue problem of a nonlinear partial differential equation under Dirichlet boundary condition in a two-dimensional space. The classical solutions are given for rectangular domains. We give numerical solutions obtained by finite element method for the first eigenvalue and eigenfunctions and we analyze the error in the approximate finite element solutions.

1. Introduction

We discuss some nonlinear eigenvalue problem arising in physics and engineering.

1.1. The vibrating membrane

Consider the small, transverse vibration of a thin membrane stretched over a bounded region D in the plane and fixed along its edges ∂D. The vertical displacement $v(x,y,t)$ of the point (x,y) in D at time t satisfies

$$-\left(v_x\,|v_x|^{p-1}\right)_x-\left(v_y\,|v_y|^{p-1}\right)_y=-\,|v|^{p-1}\,v_{tt}\,,\qquad p>0,\quad (x,y)\in D,\quad t>0, \tag{1}$$

for a positive real p, and

$$v(x,y,t)=0,\qquad (x,y)\in\partial D,\quad t\geq 0.$$

If we seek separated solutions of the form

$$v(x,y,t)=u(x,y)\,w(t),$$

in which the spatial variables x,y and the temporal variable t are separated, from (1) we are led to the nonlinear eigenvalue problem of finding λ and $u(x,y)\neq 0$ satisfying

$$-\left(u_x\,|u_x|^{p-1}\right)_x-\left(u_y\,|u_y|^{p-1}\right)_y=\lambda\,u\,|u|^{p-1}\,,\qquad p>0,\quad (x,y)\in D, \tag{2}$$

$$u(x,y)=0,\qquad (x,y)\in\partial D. \tag{3}$$

The boundary condition (3) is the Dirichlet condition. λ is called an eigenvalue and $u(x,y)$ a corresponding eigenfunction of the nonlinear eigenvalue problem, moreover (λ,u) is often called an eigenpair.

It is known, [2], that the problems of this type have a sequence of eigenvalues

$$0<\lambda_1\leq\lambda_2\leq\cdots,$$

and corresponding eigenfunctions

$$u_1(x,y),\ u_2(x,y),\ \ldots$$

Moreover it was also proved that the eigenvalues λ_j tend to ∞ as $j \to \infty$ [2].

Corresponding to each λ_j we solve

$$w''(t) + \lambda_j w(t) = 0, \qquad t > 0,$$

obtaining

$$w(t) = w_j(t) = a_j \sin \sqrt{\lambda_j}\,(t + \Theta_j),$$

where a_j and Θ_j are arbitrary.

1.2. The problem of heat conduction

Consider the problem of heat conduction in a body occupying a region D in two dimensional space. We suppose the temperature distribution through D is known at time zero, the temperature is held at zero on ∂D for all time, that we want to determine the temperature $v(x, y, t)$ at the point $(x, y) \in D$ at time $t > 0$. From the fundamental law of heat conduction of thermal orthotropic material we know that

$$\begin{aligned} &-\frac{\partial}{\partial x}\left(k_x(x,y)\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y(x,y)\frac{\partial v}{\partial y}\right) = -r(x,y)\frac{\partial v}{\partial t}, && (x,y) \in D, \quad t > 0, \\ &v(x,y,t) = 0, && (x,y) \in \partial D, \quad t \geq 0, \\ &v(x,y,0) = f(x,y), && (x,y) \in D, \end{aligned}$$

where

$f(x, y)$ – the temperature distribution at $t = 0$,

$k_x(x, y)$ and $k_y(x, y)$ – the thermal conductivity of the material along the orthotropical axes x and y,

$r(x, y)$ – density of the material times the specific heat of the material.

If we seek separated solutions

$$v(x, y, t) = u(x, y)\, w(t),$$

of the differential equation when

$$k_x(x,y) = |v_x|^{p-1}, \qquad k_y(x,y) = |v_y|^{p-1}, \qquad \text{and} \qquad r(x,y) = |v|^{p-1},$$

we are led to the eigenvalue problem (2–3) and for each eigenpair $(\lambda_j\,,\, u_j)$ of (2) we get the equation

$$w'(t) + \lambda_j w(t) = 0, \qquad t > 0. \tag{4}$$

Corresponding to each λ_j from (4) we find $w(t) = w_j(t) = a_j e^{-\lambda_j t}$. Thus the separated solutions are given by $a_j u_j(x, y) e^{-\lambda_j t}$, $j = 1, 2, \ldots$.

2. Classical solution

We say that $u(x, y)$ defined on $\bar{D} = D \cup \partial D$ is the classical solution of the eigenvalue problem (2–3) with Dirichlet boundary condition if the function u is twice differentiable in D, differentiable in $\bar{D}$, the equation (2) is satisfied for all $(x, y) \in D$ and the Dirichlet condition (3) is satisfied for all $(x, y) \in \partial D$.

If the domain D is bounded by a rectangle we have got classical solutions [1]. We seek the solution of (2) as the product of two functions of one variable

$$u(x,y) = X(x) \cdot Y(y).$$

In this case the nonlinear partial differential equation can be separated and we are led to two nonlinear ordinary differential equations. After solving them we are able to give classical solutions for (2). When D is a rectangle

$$D = \{(x,y): \ 0 \le x \le a, \ 0 \le y \le b\}$$

for the Dirichlet problem of (2)

$$\lambda_{k,l} = p\tilde{\pi}^{p+1}\left(\frac{k^{p+1}}{a^{p+1}} + \frac{l^{p+1}}{b^{p+1}}\right), \tag{5}$$

$$u_{k,l} = A_{k,l}\, S_p\left(\frac{k\tilde{\pi}}{a}x\right) S_p\left(\frac{l\tilde{\pi}}{b}y\right), \qquad k,l = 1,2,\ldots \tag{6}$$

are eigenvalues and eigenfunctions, respectively, and

$$\tilde{\pi} = \frac{2\frac{\pi}{p+1}}{\sin\frac{\pi}{p+1}},$$

$A_{k,l} = \text{const}$, and the function S_p is the solution of the differential equation

$$S_p'' \left|S_p'\right|^{p-1} + |S_p|^{p-1} S_p = 0$$

under conditions

$$S_p(0) = 0, \qquad S_p(\tilde{\pi}) = 0.$$

The function S_p is the generalized sine function introduced by A. Elbert [5]. For $p = 1$

$$S_1(x) = \sin x, \qquad \tilde{\pi} = \pi.$$

It is not known that the problem has classical solution on other domains except rectangle.

If the domain D is bounded by a unit square, as a corollary of the above, we get the smallest eigenvalue and the corresponding eigenfunction for the Dirichlet eigenvalue problem of (2) (we put $k = l = 1$ to the expressions of $\lambda_{k,l}$ and $u_{k,l}$):

$$\lambda_{1,1}(p) = 2p\tilde{\pi}^{p+1}, \qquad u_{1,1} = A_{1,1}\, S_p(\tilde{\pi}x)\, S_p(\tilde{\pi}y).$$

The first eigenvalues of a unit square are presented as a function of p in Fig. 1.

The exact value of the first eigenvalue for the linear eigenvalue problem ($p = 1$) was given by Rayleigh in his famous book titled *The Theory of Sound* [6]:

$$\lambda_{1,1}(1) = 2\pi^2 \approx 19.7392.$$

The graph of the first eigenfunction $u_{1,1}$ was presented in Fig. 2.

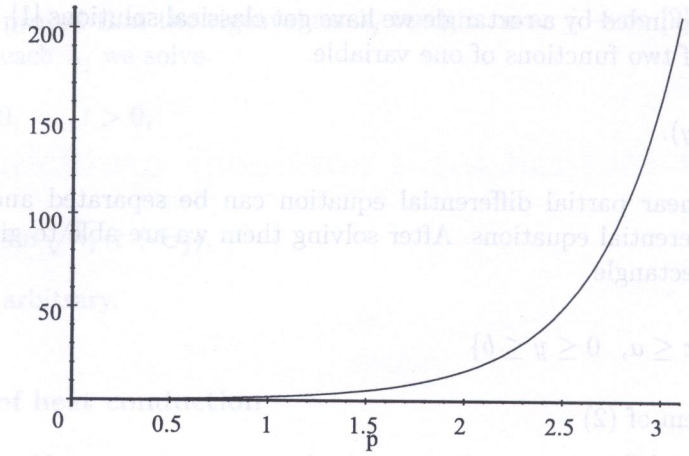

Fig. 1

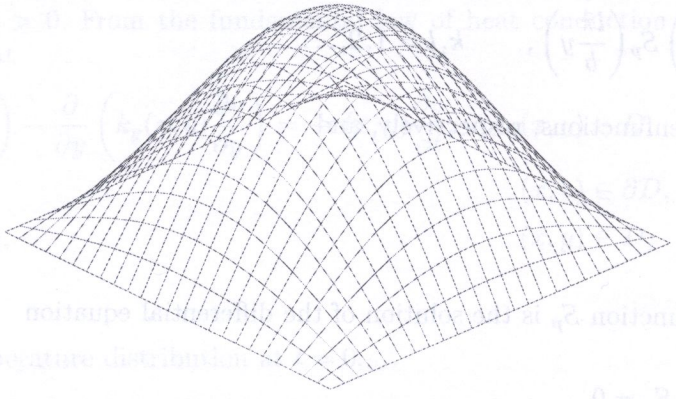

Fig. 2

3. Finite element method

We use real function spaces. Let us recall that $W^{1,p+1}(D)$, with $0 < p < \infty$, denotes the space of all functions which together with their derivatives (in the distribution sense) u_x, u_y belong to $L^{p+1}(D)$. As usual the symbol $W_0^{1,p+1}(D)$ stands for the subspace of $W^{1,p+1}(D)$ obtained by closing the set of all C^∞-functions with compact support in D, see e.g. [4].

In (2–3) the eigenvalue problem was stated in classical form, i.e., we were seeking an eigenvalue λ and a corresponding nonzero eigenfunction $u(x,y)$ such that the eigenvalue equation and boundary condition were satisfied in the classical pointwise sense. This problem can alternately be given as variational formulation:

Find $u \in W_0^{1,p+1}(D)$ such that

$$\int_D (v_x u_x |u_x|^{p-1} + v_y u_y |u_y|^{p-1})\,\mathrm{d}x = \lambda \int_D vu\,|u|^{p-1}\,\mathrm{d}x \qquad \text{for all } v \in W_0^{1,p+1}(D). \tag{7}$$

It can be shown [2] that $u \in W_0^{1,p+1}(D)$ satisfies (7) if and only if u is the solution of the following minimization problem:

Find $u \in W_0^{1,p+1}(D)$ such that

$$J(u) = \inf_{v \in W_0^{1,p+1}(D)} J(v) \qquad \text{for all } v \in W_0^{1,p+1}(D), \tag{8}$$

where the nonlinear functional $J : W_0^{1,p+1}(D) \to \mathbf{R}$ is given by

$$J(v) = \frac{1}{p+1} \int_D \left(|v_x|^{p+1} + |v_y|^{p+1} - \lambda |v|^{p+1} \right) \mathrm{d}x.$$

The basic idea in any numerical method for a differential equation is to discretize the given continuous problem with infinitely many degrees of freedom to obtain a discrete problem or system of equations with only finitely many unknowns that may be solved using a computer. For the discretization process using a finite element method we suppose that D is written as the union of linear triangular elements with maximum diameter h. To obtain a problem that can be solved on a computer the idea in the finite element method is to replace $W_0^{1,p+1}(D)$ by a set V_h consisting of simple functions only depending on finitely many parameters. The functions N_i, $i = 1, 2, \ldots, m$ are piecewise linear for the m dimensional subspace V_h of $W_0^{1,p+1}(D)$, thus

$$V_h = \left\{ \sum_{i=1}^{m} x_i N_i \,|(x_1\,, x_2\,, \ldots\,, x_m) \in \mathbf{R}^m \right\}.$$

This leads to a finite-dimensional minimization problem of the form:

Find $u_h \in V_h$, $\lambda \in \mathbf{R}$ such that

$$J(u_h) = \inf_{v_h \in V_h} J(v_h) \qquad \text{for all } \; v_h \in V_h\,. \tag{9}$$

Since $V_h \subset W_0^{1,p+1}(D)$ then (9) corresponds to the classical Ritz–Galerkin method. In the finite element method as a particular Ritz–Galerkin method the functions in V_h are chosen to be piecewise linear.

The minimization problem (8) corresponds to the fundamental principle of minimum potential energy in mechanics. It can be shown that the solution of the differential equation (2) also is a solution of (8).

The finite element approximation of u is

$$u_h = \sum_{i=1}^{m} x_i N_i = \mathbf{x}^T \mathbf{N}^T,$$

and

$$\nabla u_h = \sum_{i=1}^{m} \nabla (N_i x_i) = \mathbf{x}^T \mathbf{B}^T.$$

Hence we get the eigenvalue of the problem (2) from the formula

$$\lambda_h = \frac{\displaystyle\int_D \left(|(u_h)_x|^{p+1} + |(u_h)_y|^{p+1} \right) \mathrm{d}x}{\displaystyle\int_D |u_h|^{p+1}\, \mathrm{d}x}\,, \tag{10}$$

where u_h are the finite element approximations of the eigenfunctions. The first eigenvalue we obtain as

$$\lambda_{1h} = \inf_{u \in V_h} \frac{\displaystyle\int_D \left(|u_x|^{p+1} + |u_y|^{p+1} \right) \mathrm{d}x}{\displaystyle\int_D |u|^{p+1}\, \mathrm{d}x}\,.$$

Let (λ_{1h}, u_{1h}) with

$$\|u_h\| = \left(\int_D |u|^{p+1}\,dx\right)^{\frac{1}{p+1}} = 1,$$

be the normalized eigenpair solving (9). It is known [3] that

$$|\lambda_{1h} - \lambda_1| \to 0, \qquad |u_{1h} - u_1| \to 0 \qquad \text{as} \quad h \to 0,$$

where h is the finite element mesh parameter of V_h.

To compute λ_{1h} and u_{1h} we use the algorithm:

i. Normalization. Let

$$\int_D |u_h|^{p+1}\,dx = 1, \qquad \text{that is} \qquad \int_D |\mathbf{N}\,\mathbf{x}|^{p+1}\,dx = 1.$$

For arbitrary $\tilde{u}_h = \mathbf{N}\tilde{\mathbf{x}} = \mathbf{N}(\mathbf{x}a)$, where $a \neq 0$, we have

$$\int_D |\tilde{u}_h|^{p+1}\,dx = \int_D |\mathbf{N}\mathbf{x}a|^{p+1}\,dx = \int_D |\mathbf{N}\mathbf{x}|^{p+1}|a|^{p+1}\,dx,$$

therefore

$$|a| = \left[\int_D |\tilde{u}_h|^{p+1}\,dx\right]^{\frac{1}{p+1}} \qquad \text{and} \qquad \mathbf{x} = \frac{\tilde{\mathbf{x}}}{a}.$$

ii. After normalization we obtain from (10) that

$$\lambda_h = \int_D \left(\left|\frac{\partial}{\partial x}\mathbf{N}\mathbf{x}\right|^{p+1} + \left|\frac{\partial}{\partial y}\mathbf{N}\mathbf{x}\right|^{p+1}\right) dx = \int_D |\mathbf{B}\mathbf{x}|^{p+1}\,dx \tag{11}$$

and λ_{1h} is the minimizer of (10). From $\frac{\partial \lambda_h}{\partial x_i} = 0$, $(i = 1, 2, \ldots, m)$ we get

$$0 = \left[\int_D \mathbf{B}^T |\mathbf{B}\mathbf{x}|^p\,dx\right]\left[\int_D |\mathbf{N}\mathbf{x}|^{p+1}\,dx\right] - \left[\int_D |\mathbf{B}\mathbf{x}|^{p+1}\,dx\right]\left[\int_D \mathbf{N}^T |\mathbf{N}\mathbf{x}|^p\,dx\right]$$

and using (11) we obtain

$$0 = \int_D \mathbf{B}^T |\mathbf{B}\mathbf{x}|^p\,dx - \lambda_h \int_D \mathbf{N}^T |\mathbf{N}\mathbf{x}|^p\,dx. \tag{12}$$

Since (12) is nonlinear we get

$$0 = \int_D \mathbf{B}^T |\mathbf{B}\mathbf{x}_j|^p\,dx - \lambda_{h,j} \int_D \mathbf{N}^T |\mathbf{N}\mathbf{x}_j|^p\,dx + p\int_D \mathbf{B}^T |\mathbf{B}\mathbf{x}_j|^{p-1}\,\mathbf{B}\,dx\,(\mathbf{x} - \mathbf{x}_j) \tag{13}$$

by using the Newton's method for given $\mathbf{x}_j$. Applying the following notations

$$\mathbf{H}_j = p\int_D \mathbf{B}^T |\mathbf{B}\mathbf{x}_j|^{p-1}\,\mathbf{B}\,dx,$$

$$\mathbf{f}_N = \int_D \mathbf{N}^T |\mathbf{N}\mathbf{x}_j|^p\,dx,$$

$$\mathbf{f}_B = \int_D \mathbf{B}^T |\mathbf{B}\mathbf{x}_j|^p\,dx,$$

$$\Delta\mathbf{x}_j = (\mathbf{x} - \mathbf{x}_j),$$

and we can write (13) as follows

$$\mathbf{H}_j\,\Delta\mathbf{x}_j = \lambda_{h,j}\mathbf{f}_N - \mathbf{f}_B.$$

We briefly describe the algorithm below:

1. Start with an initial approximation $\mathbf{x}_1$ ($\mathbf{x}_1 \neq \mathbf{0}$).
2. For given $\mathbf{x}_j$, calculate $\mathbf{H}_j \Delta \mathbf{x}_j = \lambda_{h,j} \mathbf{f}_N - \mathbf{f}_B$, then $\Delta \mathbf{x}_j$.
3. Evaluate $\tilde{\mathbf{x}}_{j+1} = \mathbf{x}_j + \Delta \mathbf{x}_j$.
4. Evaluate $\mathbf{x}_{j+1} = \dfrac{\tilde{\mathbf{x}}_{j+1}}{\left[\int_D |\mathbf{N}\mathbf{x}_{j+1}|^{p+1}\, \mathrm{d}x\right]^{\frac{1}{p+1}}}$.
5. Calculate $\lambda_{h,j+1} = \displaystyle\int_D |\mathbf{B}\mathbf{x}_{j+1}|^{p+1}\, \mathrm{d}x$.
6. Terminate when $\dfrac{|\lambda_{h,j+1} - \lambda_{h,j}|}{\lambda_{h,j+1}}$ is smaller then a predetermined tolerance.

4. Numerical results

Our computations in this paper were performed for the unit square. Here the exact values are known as a function of p. We approximated λ_1 for various values of p using triangular elements with dividing the sides of the square into 2, 4, 8 or 16 equal parts. The numerical results obtained by the finite element method are reported in Table 1. The tolerance value was 10^{-5} in the presented calculations. We have got different number of iterations for different values of p. The least iteration numbers (5–7 iterations) were obtained in the linear ($p = 1$) case. For $p = 0.25$ we got 39–594 iterations and 10–22 iterations for $p = 9$.

These results will be compared with the classical solutions obtained in Part 2 (see Table 2).

The knowledge of classical solutions makes it possible to analyze the error in the approximate finite element solution. In Table 1. the relative error is calculated for different mesh sizes and for different values of p. We show the relative error in the first eigenvalues when using different mesh size, when the number of nodal points N are different. In Fig. 3 the logarithm of the relative error is presented for different values of $\log N$ when $p = 2$. We see that to get relatively high accuracy for the eigenvalues is not difficult. In our case, with 16 equal parts on the sides of the square, we already have an accuracy of 1%.

Table 1

p	2	E(%)	4	E(%)	8	E(%)	16	E(%)
0.25	10.7316	46.76	7.82924	7.07	7.4349	1.68	7.3459	0.46
0.5	14.1856	33.36	11.1788	5.09	10.7632	1.18	10.6683	0.29
0.75	18.5345	26.21	15.4269	5.05	14.8608	1.19	14.7285	0.29
1	24.0000	21.58	20.7733	5.24	19.9942	1.29	19.8027	0.32
1.25	30.8662	18.33	27.4956	5.41	26.4513	1.40	26.1785	0.36
1.5	39.4950	15.97	35.9374	5.52	34.5713	1.51	34.1917	0.39
2	64.0000	13.12	59.7529	5.61	57.5336	1.79	56.8420	0.47
3	164.571	12.63	154.144	5.49	148.924	1.92	146.96	0.58
4	418.909	17.24	376.421	5.34	364.646	2.05	359.712	0.67
5	1063.38	25.99	890.323	5.48	862.019	2.13	850.263	0.74
6	2697.36	38.61	2064.90	6.11	1988.77	2.19	1961.44	0.79
7	6840.74	55.22	4729.90	7.33	4507.16	2.27	4443.81	0.83
8	17347.7	76.25	10747.7	9.20	10077.1	2.38	9928.48	0.87
9	43992.4	102.4	24293.4	11.74	22294.1	2.55	21937.7	0.91

Table 2

p	$\lambda_1(p)$
0.25	7.31196
0.5	10.63743
0.75	14.68546
1	**19.7392**
1.25	26.08514
1.5	34.05689
2	56.57752
3	146.1136
4	357.3105
5	844.0179
6	1945.989
7	4406.865
8	9842.174
9	21739.39

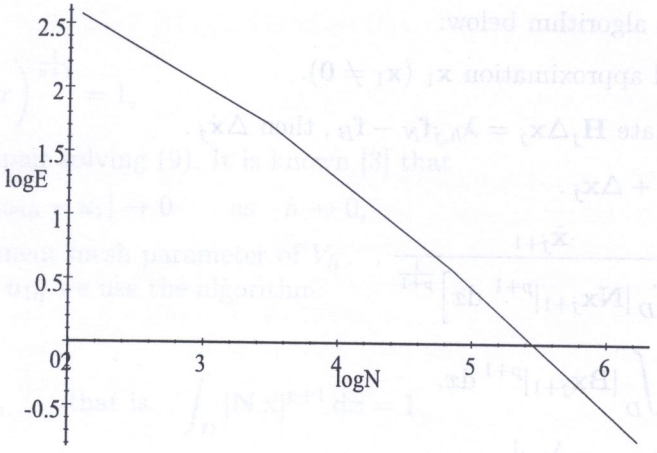

Fig. 3

ACKNOWLEDGEMENTS

- The author is grateful to T. Szabó (Department of Mechanics, University of Miskolc) for his valuable comments and his personal help in the computations.
- This work was supported by the Hungarian National Research Foundation OTKA T026138.

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