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Trefftz functions as basic functions of FEM in application to solution of inverse heat conduction problem

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The work presents the application of heat polynomials for solving an inverse problem. The heat polynomials form the Trefftz Method for non-stationary heat conduction problem. They have been used as base functions in Finite Element Method. Application of heat polynomials permits to reduce the order of numerical integration as compared to the classical Finite Element Method with formulation of the matrix of system of equations.

1. Introduction

Heat polynomials have been presented for the first time in [5], and their application is reported in [1, 2, 3, 6]. These polynomials solve the heat conduction equation, and their linear combination is used to solve non-linear stationary heat conduction problems by the Trefftz method. Completeness of their polynomial basis ensures a very good approximation of the temperature function in a finite element. The essence of the work presented here consists in introducing space–time elements and in defining a functional for the solution of the non-stationary heat conduction equation by the Finite Element Method implemented on a heat polynomial basis. The resulting stiffness matrix is still symmetric and positive definite but its dimension of integration is reduced by one order.

The calculation of a boundary condition based on the measurement of the temperature at a point inside the domain is important in many applications. This inverse problem is solved here with the use of heat polynomials. An inverse problem of non-stationary heat conduction in a planar layer is solved to investigate the numerical properties of heat polynomials.

2. Governing equation and solution of simple problem

Consider the one-dimensional linear heat conduction equation for a layer

$$\rho \cdot c \frac{\partial T}{\partial \tau} = \lambda \frac{\partial^2 T}{\partial x^2}, \qquad \tau \in (0, \infty), \quad x \in (0, l), \quad T \in C^2(0,l) \cap C^1(0,\infty), \tag{1}$$

with constant coefficients ρ, c, λ and the following conditions:

- initial condition

$$T(x, 0) = f(x), \qquad x \in \langle 0, l \rangle, \tag{2}$$

- boundary condition (Fig. 1)

$$T(x{=}0, \tau) = g(\tau), \qquad \tau \in (0, \infty), \tag{3}$$

$$T(x{=}1, \tau) = h(\tau), \qquad \tau \in (0, \infty). \tag{4}$$

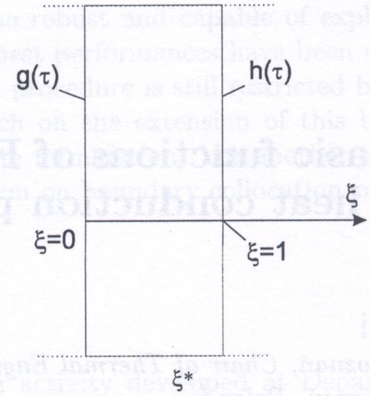


Fig. 1

The auxiliary variables (5) are used to write the heat conduction equations in the non-dimensional form (6),

$$\xi = \frac{x}{l}\,, \qquad t = \frac{\lambda}{\rho \cdot c} \cdot \frac{\tau}{l^2}\,, \tag{5}$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial \xi^2}\,, \qquad \xi \in (0,1), \quad t \in (0,\infty). \tag{6}$$

The following result is obtained by expanding the function $T(\xi,t) \in C^\infty$ in a Taylor series on point $(\xi_o\,, t_o)$ and eliminating even derivatives on ξ using Eq. (6),

$$\begin{aligned}
T(x,t) &= T_o \cdot 1 \\
&+ \frac{\partial T}{\partial x} \cdot \bar{x} + \frac{\partial T}{\partial t}\bar{t} \\
&+ \frac{\partial T}{\partial t} \cdot \frac{\bar{x}^2}{2!} + \frac{\partial^2 T}{\partial x \partial t}\bar{x}\bar{t} + \frac{\partial^2 T}{\partial t^2} \cdot \frac{\bar{t}^2}{2!} \\
&+ \frac{\partial^2 T}{\partial x \partial t} \cdot \frac{\bar{x}^3}{3!} + \frac{\partial^2 T}{\partial t^2} \cdot \frac{\bar{x}^2\bar{t}}{2!} + \frac{\partial^3 T}{\partial x \partial t^2} \cdot \frac{\bar{x}\bar{t}^2}{2!} + \frac{\partial^3 T}{\partial t^3} \cdot \frac{\bar{t}^3}{3!} \\
&+ \frac{\partial^2 T}{\partial t^2} \cdot \frac{\bar{x}^4}{4!} + \frac{\partial^3 T}{\partial x \partial t^2} \cdot \frac{\bar{x}^3\bar{t}}{3!} + \frac{\partial^3 T}{\partial t^3} \cdot \frac{\bar{x}^2\bar{t}^2}{2!2!} + \frac{\partial^4 T}{\partial x \partial t^3} \cdot \frac{\bar{x}\bar{t}^3}{3!} + \frac{\partial^4 T}{\partial t^4} \cdot \frac{\bar{t}^4}{4!} \\
&+ \frac{\partial^3 T}{\partial x \partial t^2} \cdot \frac{\bar{x}^5}{5!} + \frac{\partial^3 T}{\partial t^3} \cdot \frac{\bar{x}^4\bar{t}}{4!} + \frac{\partial^4 T}{\partial x \partial t^3} \cdot \frac{\bar{x}^3\bar{t}^2}{3!2!} + \frac{\partial^4 T}{\partial t^4} \cdot \frac{\bar{x}^2\bar{t}^3}{2!3!} + \frac{\partial^5 T}{\partial x \partial t^4} \cdot \frac{\bar{x}\bar{t}^4}{4!} + \frac{\partial^5 T}{\partial t^5} \cdot \frac{\bar{t}^5}{5!} \\
&+ \frac{\partial^3 T}{\partial t^3} \cdot \frac{\bar{x}^6}{6!} + \frac{\partial^4 T}{\partial x \partial t^3} \cdot \frac{\bar{x}^5\bar{t}}{5!} + \frac{\partial^4 T}{\partial t^4} \cdot \frac{\bar{x}^4 t^2}{4!2!} + \frac{\partial^5 T}{\partial x \partial t^4} \cdot \frac{\bar{x}^3\bar{t}^3}{3!3!} + \frac{\partial^5 T}{\partial t^5} \cdot \frac{\bar{x}^4\bar{t}^2}{4!2!} + \frac{\partial^6 T}{\partial x \partial t^5} \cdot \frac{\bar{x}\bar{t}^5}{5!} + \frac{\partial^6 T}{\partial t^6} \cdot \frac{\bar{t}^6}{6!} \\
&+ \frac{\partial^4 T}{\partial x \partial t^3} \cdot \frac{\bar{x}^7}{7!} + \ldots + R_n
\end{aligned} \tag{7}$$

$$T_o = T(x_o\,, t_o), \qquad \bar{x} = x - x_0\,, \qquad \bar{t} = t - t_0\,.$$

It is noted that the derivatives of equal order with regard to the same variables are located at the diagonal, from the left to the right side in Eq. (7). Hence, grouping of the terms of the expansion (7) with the same derivatives yields the following solution for Eq. (6).

$$T(x,t) = T(x_o\,,\,t_o)\cdot 1 + \frac{\partial T}{\partial x}\cdot \bar{x} + \frac{\partial T}{\partial t}\left(\frac{\bar{x}^2}{2!}+\bar{t}\right) + \frac{\partial^2 T}{\partial x \partial t}\left(\frac{\bar{x}^3}{3!}+\bar{x}\cdot\bar{t}\right) + \frac{\partial^2 T}{\partial t^2}\left(\frac{\bar{x}^4}{4!}+\frac{\bar{x}^2 t}{2!}+\frac{\bar{t}^2}{2!}\right)$$
$$+\frac{\partial^3 T}{\partial x \partial t^2}\left(\frac{\bar{x}^5}{5!}+\frac{\bar{x}^3 t}{3!}+\frac{\bar{x}\bar{t}^2}{2!}\right) + \frac{\partial^3 T}{\partial t^3}\left(\frac{\bar{x}^6}{6!}+\frac{\bar{x}^4 t}{4!}+\frac{\bar{x}^2\bar{t}^2}{2!2!}+\frac{\bar{t}^3}{3!}\right)$$
$$+\frac{\partial^4 T}{\partial x \partial t^3}\left(\frac{\bar{x}^7}{7!}+\frac{\bar{x}^5 t}{5!}+\frac{\bar{x}^3\bar{t}^2}{3!2!}+\frac{\bar{x}\bar{t}^3}{3!}\right) + S_n \tag{8}$$

The polynomials in expansion (8) solve Eq. (6). Solution (8) can be written in the following form,

$$T(\xi,t) = \sum_{n=0}^{\infty}\frac{\partial^n T}{\partial t^n}\cdot v_{2n}(\bar{\xi},\bar{t}) + \sum_{n=0}^{\infty}\frac{\partial^{n+1} T}{\partial \xi \partial t^n}\cdot v_{2n+1}(\bar{\xi},\bar{t}) = \sum_{n=0}^{\infty}\frac{\partial^n T}{\partial \xi^n}\cdot v_n(\bar{\xi},\bar{t}), \tag{9}$$

where

$$v_n(\bar{\xi},\bar{t}) = \sum_{k=0}^{[\frac{n}{2}]}\frac{\bar{\xi}^{n-2k}}{(n-2k)}\cdot\frac{\bar{t}^k}{k!} \tag{10}$$

The univocal character of the Taylor expansion of function $T(\xi,t)$ gives

$$T(\xi,t) - \sum_{n=0}^{N}\frac{\partial^n T(\xi_o,t_o)}{\partial \xi^n}\cdot v_n(\xi,t) = S_{N+1} \quad \text{and} \quad \lim_{N\to\infty} S_N = 0 \tag{11}$$

meaning that the system of functions $\{v_o\,, v_1\,, \ldots, v_N\,, \ldots\}$ is complete.

Functions $v_n(\xi,t)$ are called heat polynomials [5]. The determination of values of function $v_n(\xi,t)$ using expression (10) may be charged with considerable numerical errors. The values of function $v_n(\xi,t)$ can be determined by recurrence formulae. In order to derive the recurrence formulae one should notice that the function $\mathbf{W} = \exp(p\xi^2 + p^2 t)$ solves Eq. (6) for any value of parameter p. The generating function, $\mathbf{W}$, can be expanded into a power series with regard to parameter p,

$$W = e^{p\xi + p^2 t} = \sum_{n=0}^{\infty} v_n(\xi,t)\cdot p^n, \qquad p\in R. \tag{12}$$

Expansion of the generating function $\mathbf{W}$ leads to the following relations,

$$v_o(\xi,t) = 1, \qquad v_1(\xi,t) = \xi, \qquad v_2(\xi,t) = \frac{1}{2}\xi^2 + t, \qquad v_3(\xi,t) = \frac{1}{6}\xi^3 + t\cdot\xi,$$
$$v_4(\xi,t) = \frac{1}{24}\xi^4 + \frac{1}{2}\xi^2 t + \frac{1}{2}t^2, \qquad v_5(\xi,t) = \frac{1}{120}\xi^5 + \frac{1}{6}\xi^3 t + \frac{1}{2}\xi t^2,$$
$$v_6(\xi,t) = \frac{1}{720}\xi^6 + \frac{1}{24}\xi^4 t + \frac{1}{4}\xi^2 t^2 + \frac{1}{6}t^3, \qquad v_7(\xi,t) = \frac{1}{5040}\xi^7 + \frac{1}{120}\xi^5 t + \frac{1}{12}\xi^3 t^2 + \frac{1}{6}\xi t^3,$$
$$\vdots$$
$$v_{n+1}(\xi,t) = \frac{\xi}{n}\cdot v_n(\xi,t) + \frac{2t}{n}\cdot v_{n-1}(\xi,t), \qquad n\geq 1. \tag{13}$$

The following relationships are satisfied for the derivatives,

$$\frac{\partial v_n(\xi,t)}{\partial \xi} = v_{n-1}(\xi,t), \quad n\geq 1, \qquad \frac{\partial v_n(\xi,t)}{\partial t} = v_{n-2}(\xi,t), \quad n\geq 2. \tag{14}$$

The polynomials $v_n(\xi,t)$ identically satisfy the heat conduction equation (6). The solution of the heat conduction equation can be presented in the form

$$T(\xi,t) = \sum_{n=0}^{\infty} \frac{\partial^n T}{\partial \xi^n} \cdot v_n(\bar{\xi},\bar{t}) = \sum_{n=1}^{\infty} A_n \cdot v_n(\bar{\xi},\bar{t}). \tag{15}$$

The unknown coefficients, A_n , in the approximate solution,

$$T(\xi,t) \approx \sum_{n=0}^{N-1} A_n \cdot v_n(\bar{\xi},\bar{t}) = \{A\}^T \cdot \{v\}, \tag{16}$$

are sought through the minimization of the mean square distance of solution (16) from the initial and boundary conditions. The functional form of this distance is as follows,

$$J(\{A\}) = \int_0^1 [T(\xi,0) - f(\xi)]^2 \, \mathrm{d}\xi + \int_0^{t_k} [T(0,\tau) - g(\tau)]^2 \, \mathrm{d}\tau + \int_0^{t_k} [T(1,\tau) - h(\tau)]^2 \, \mathrm{d}\tau. \tag{17}$$

It is noted that at the initial time $t = 0$ ($t_o = 0$)

$$T(\xi,0) = \sum_{n=0}^{N-1} A_n \cdot v_n(\xi - \xi_o\,,\, 0) = \sum_{n=0}^{N-1} A_n \frac{(\xi - \xi_o)^n}{n!} = \sum_{n=0}^{N-1} A_n \frac{\bar{\xi}^n}{n!}$$

Hence, the initial condition $f(\xi)$ is approximated by linear combination of functions 1, $\bar{\xi}$, $\bar{\xi}^2$, $\bar{\xi}^3$, ..., that leads to Hilbert matrix elements in the matrix of the coefficients $\{A\}$. Therefore, stable numerical results can be obtained for $N \approx 12$. In the case of rapid heating or cooling of bodies, the large gradients of the function ξ that occur require the number N of heat polynomials in approximation (16) to be increased. This increase leads to ill-conditioning of the system of equations defining coefficients $\{A\}$. A solution of this problem consists in dividing the solution in finite elements and approximating the solution of Eq. (6) in the form of linear combination of heat functions (16).

Division of the solution range in the elements is shown in Fig. 2. Physical properties of heat conduction process result in the fact that both temperature and heat flux at common element boundaries are continuous functions. Due to a finite linear combination (16) of the solution approximation one may require continuity of temperature and possibly small differences in heat fluxes $q_{j-1} - q_j$ at the common boundary. Figure 3 shows an approximation of the function $T(\xi,t)$ in the

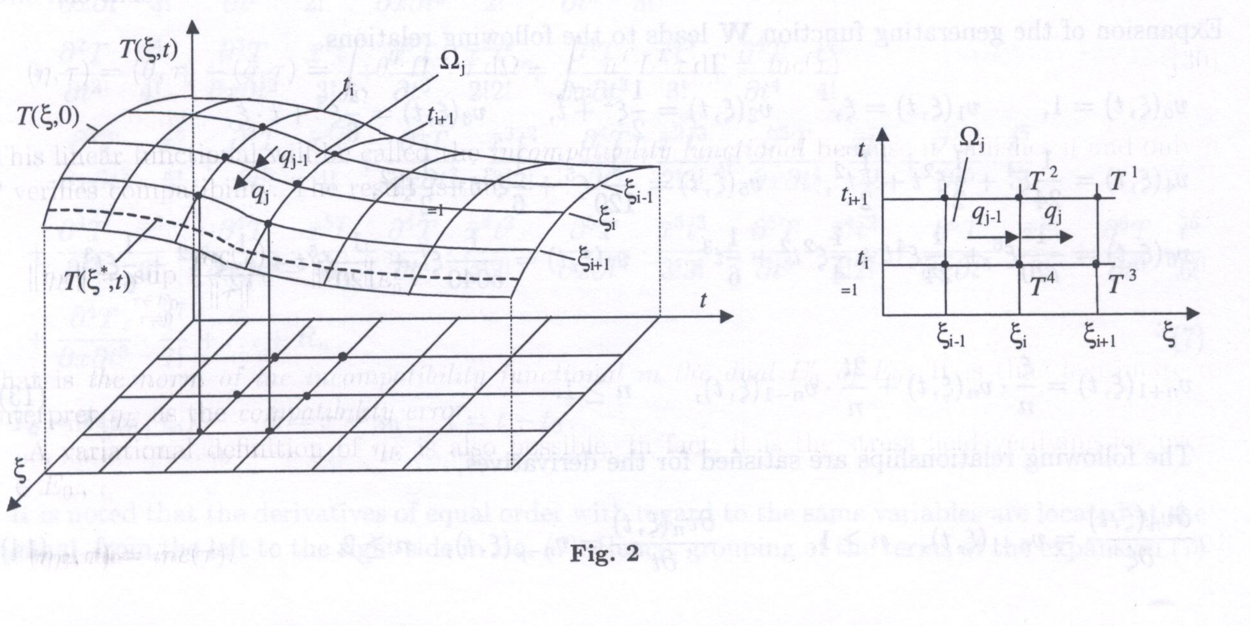


Fig. 2

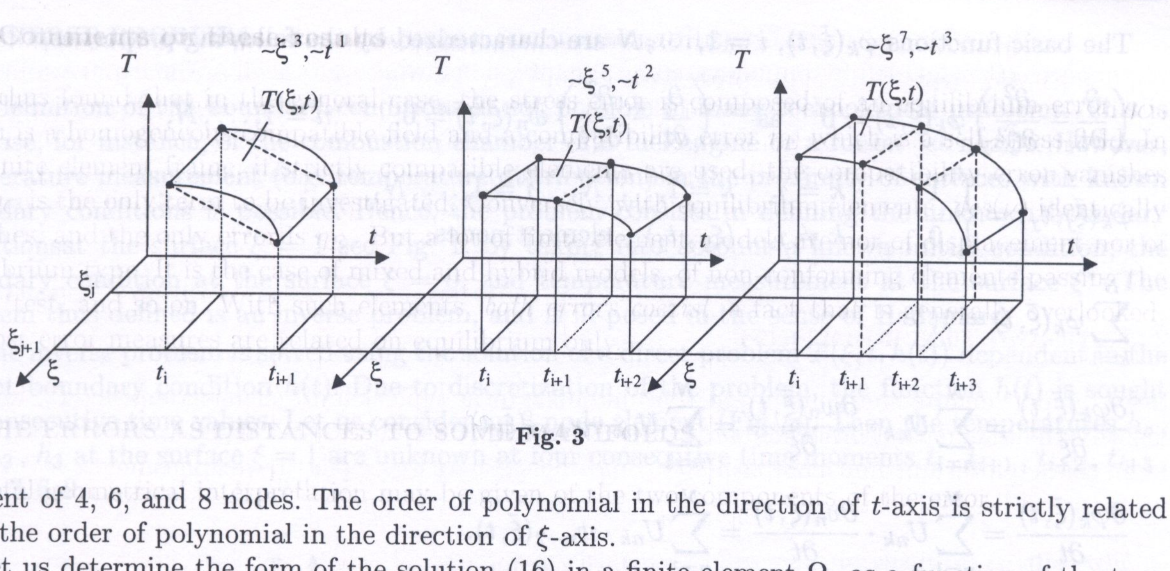


Fig. 3

element of 4, 6, and 8 nodes. The order of polynomial in the direction of t-axis is strictly related with the order of polynomial in the direction of ξ-axis.

Let us determine the form of the solution (16) in a finite element Ω_j as a function of the temperature at the finite element nodes. The temperature function is expressed by the relationship

$$T_j(\bar{\xi},t) = \sum_{n=1}^{N} A_n^j \cdot w_n(\bar{\xi},t), \qquad (\bar{\xi},t) \in \Omega_j\,, \qquad \bar{\xi} = \xi - \xi_j\,, \qquad \xi_j \le \xi \le \xi_{j+1}\,, \tag{18}$$

where for simplicity of programming the condition $w_n = v_{n-1}$ has been assumed.

Function $w_n(\bar{\xi},t)$ satisfies also the following equation,

$$\frac{\partial w_n}{\partial t} = \frac{\partial^2 w_n}{\partial \bar{\xi}^2}\,, \qquad w_n = v_{n-1}\,, \quad n = 1, 2, \ldots$$

The solution at the finite element nodes,

$$T(\bar{\xi}_k\,, t_k) = T_k = \sum_{n=1}^{N} A_n \cdot w_n(\bar{\xi}_k\,, t_k), \qquad k = 1, 2, \ldots, N$$

provides the system of equations used to define the coefficients $\{A_n\}$ for each element,

$$\begin{bmatrix} w_1(\bar{\xi}_1\,, t_1) & w_2(\bar{\xi}_1\,, t_1) & w_3(\bar{\xi}_1\,, t_1) & \cdots & w_N(\bar{\xi}_1\,, t_1) \\ w_1(\bar{\xi}_2\,, t_2) & w_2(\bar{\xi}_2\,, t_2) & w_3(\bar{\xi}_2\,, t_2) & \cdots & w_N(\bar{\xi}_2\,, t_2) \\ w_1(\bar{\xi}_3\,, t_3) & w_2(\bar{\xi}_3\,, t_3) & w_3(\bar{\xi}_3\,, t_3) & \cdots & w_N(\bar{\xi}_3\,, t_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1(\bar{\xi}_N\,, t_N) & w_2(\bar{\xi}_N\,, t_N) & w_3(\bar{\xi}_N\,, t_N) & \cdots & w_N(\bar{\xi}_N\,, t_N) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_N \end{Bmatrix} = \begin{Bmatrix} T^1 \\ T^2 \\ T^3 \\ \vdots \\ T^N \end{Bmatrix}. \tag{19}$$

The solution of this equation,

$$[w]\{A\} = \{T\}, \tag{20}$$

$$\{A\} = [w]^{-1}\{T\} = [U]\{T\}, \qquad A_n = \sum_{k=1}^{N} U_{nj} \cdot T^k,$$

yields the following results,

$$T(\xi,t) = \sum_{n=1}^{N}\left(\sum_{k=1}^{N} U_{nk} \cdot T^k\right) w_n(\bar{\xi},t) = \sum_{k=1}^{N}\left(\sum_{n=1}^{N} U_{nk} \cdot w_n(\bar{\xi},t)\right) \cdot T^k = \sum_{k=1}^{N} \varphi_k(\bar{\xi},t) \cdot T^k \tag{21}$$

$$\varphi_k(\bar{\xi},t) = \sum_{n=1}^{N} U_{nk} \cdot w_n(\bar{\xi},t), \qquad t \in \langle t_i\,, t_{i+k}\rangle, \qquad k = (N-2)/2.$$

The basic functions $\varphi_k(\bar{\xi},t)$, $i=1,\dots,N$ are characterized by the following properties,

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial\bar{\xi}^2}\right)\varphi_k(\bar{\xi},t)=0 \quad \text{as} \quad \left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial\bar{\xi}^2}\right)w_n(\bar{\xi},t)=0, \qquad n=1,\dots,N, \tag{22}$$

$$\varphi_k(\bar{\xi}_j,t_j)=\begin{cases}1 & k=j, \quad k=1,\dots,N, \quad j=1,\dots,N,\\ 0 & k\neq j, \quad (\bar{\xi}_j\,,\,t_j)\text{ – element nodes,}\end{cases} \tag{23}$$

$$\sum_{k=1}^{N}\varphi_k(\bar{\xi},t)=1, \tag{24}$$

$$\frac{\partial\varphi_k(\bar{\xi},t)}{\partial\bar{\xi}}=\sum_{n=1}^{N}U_{nk}\cdot\frac{\partial w_n(\bar{\xi},t)}{\partial\bar{\xi}}=\sum_{n=1}^{N}U_{nk}\cdot w_{n-1}(\bar{\xi},t), \tag{25}$$

$$\frac{\partial\varphi_k(\bar{\xi},t)}{\partial t}=\sum_{n=1}^{N}U_{nk}\cdot\frac{\partial v_n(\bar{\xi},t)}{\partial t}=\sum_{n=1}^{N}U_{nk}\cdot w_{n-2}(\bar{\xi},t). \tag{26}$$

The defect of the heat flux flowing between the elements caused by the approximation process is defined in [1] by

$$\delta\dot{q}_j(t)=-\frac{\partial T_j(\xi_{j+1}\,,\,t)}{\partial\xi}+\frac{\partial T_{j+1}(\xi_{j+1}\,,\,t)}{\partial\xi}=\dot{q}_j-\dot{q}_{j+1}\,. \tag{27}$$

It provides a basis for formulation of a functional of heat flux error on the boundary between elements, Fig. 4. The functional is a sum of heat flux defects at the element boundaries and takes the form

$$I_q=\sum_{j=1}^{L}\int_{t_i}^{t_{i+k}}[\delta\dot{q}_j(\tau)]^2\,\mathrm{d}\tau=\sum_{j=1}^{L}\int_{t_i}^{t_{i+k}}[\dot{q}_j(\tau)-\dot{q}_{j+1}(\tau)]^2\,\mathrm{d}\tau, \qquad k=N/2-1. \tag{28}$$

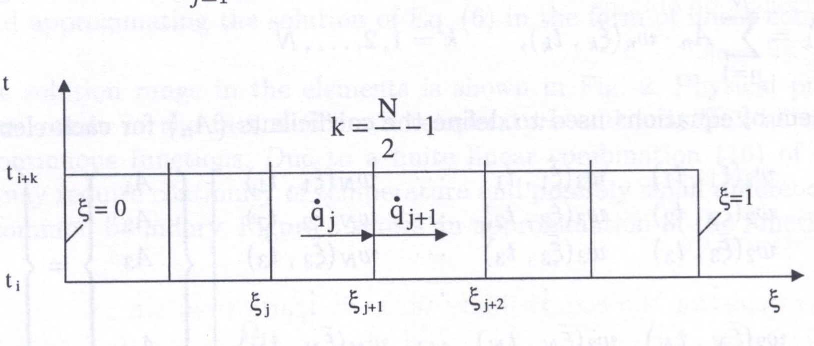


Fig. 4

The unknown temperatures at the nodes of finite element mesh are determined by minimizing the expanded functional (29) as compared to (17),

$$J_i=\sum_{l=1}^{L}\int_{x_2}^{x_{l+1}}[T_l(\xi,t_i)-T_l^o(\xi,t_i)]^2\,\mathrm{d}\xi+\int_{t_i}^{t_{i+k}}[T_1(0,\tau)-g(\tau)]^2\,\mathrm{d}\tau+\int_{t_i}^{t_{i+k}}[T_L(1,\tau)-h(\tau)]\,\mathrm{d}\tau$$

$$+\sum_{j=1}^{L}\int_{t_i}^{t_{i+k}}[\dot{q}_j(\tau)-\dot{q}_{j+1}(\tau)]^2\,\mathrm{d}\tau, \qquad i=0,k,2k,\dots; \qquad k=N/2-1, \tag{29}$$

where $T_l^o(\xi,t_i)$ is the initial temperature for time t_i.

The minimization of functional (29) leads to a solution at each mesh node, which depends on the initial and boundary conditions assumed.

3. Inverse problem

The definition of the boundary conditions is not possible in many technological problems. This is the case, for instance, of the combustion chamber of a fuel engine or a rocket jet nozzle. However, temperature measurement (e.g. temperature distribution) in the proximity of surfaces with known boundary conditions is possible. Hence, the problem consists in defining the unknown boundary conditionsat the surface $\xi = 1$ see Fig. 1, by taking into account a known initial condition, the boundary condition at the surface $\xi = 0$, and temperature measurement at the surface ξ^*. The problem thus defined is an inverse problem, and is ill-posed in the sense of Hadamard.

The inverse problem is solved using the solution of a direct problem $T(\xi, t, h(t))$ dependent in the sought boundary condition $h(t)$. Due to discretization of the problem, the function $h(t)$ is sought for consecutive time values. Let us consider an 8-node element (Fig. 5). Then the temperatures h_o, h_1, h_2, h_3 at the surface $\xi = 1$ are unknown at four consecutive time moments t_i, t_{i+1}, t_{i+2}, t_{i+3}, $i = 0, 3, 6, 9, \ldots$

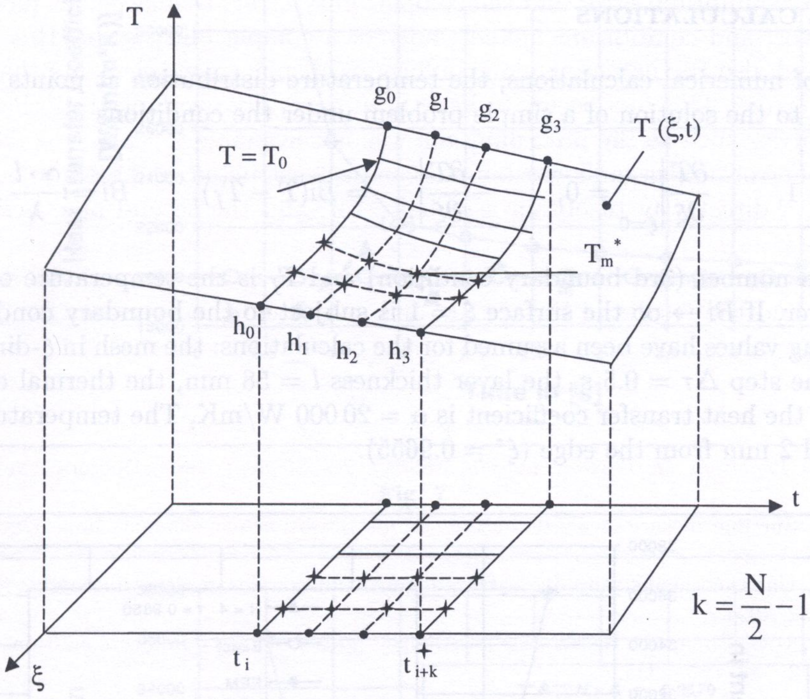


Fig. 5. Solution in time layer $\langle t_i, t_{i+k} \rangle$

Let us assume that temperature values $T^* = T(\xi^*, t_k^*)$ at point ξ^* are known at time t_k^*, $k \geq 4$. The boundary condition vector $\{h\}$ is determined from minimization of the following functional,

$$J_E = \|T(\xi^*, t^*, \{h\}) - T^*\|^2 . \tag{30}$$

The minimization of the functional (30) leads to the solution of the inverse problem in the form

$$\{T\} = [\mathrm{STAB}]\{T_o\} + \sum_{m=1}^{M} Z_m \cdot T_m^* + \{\mathrm{WA}\} \cdot g_o + \{\mathrm{WB}\} \cdot g_1 + \{\mathrm{WC}\} \cdot g_2 + \{\mathrm{WD}\} \cdot g_3 \tag{31}$$

The temperature disturbance T_m^* at point (ξ_m, t_m), δT_m^*, results in disturbance of temperature T_o, δT_o (beside the first time-layer). So, the disturbed temperature distribution may be expressed by the relationship

$$\{T+\delta T\} = [\mathrm{STAB}]\cdot\{T_o+\delta T_o\} + \sum_{m=1}^{M} Z_m\cdot(T_m^* + \delta T_m^*)$$
$$+ \{\mathrm{WA}\}\cdot g_o + \{\mathrm{WB}\}\cdot g_1 + \{\mathrm{WC}\}\cdot g_2 + \{\mathrm{WD}\}\cdot g_3\,, \tag{32}$$
$$\{\delta T\} = [\mathrm{STAB}]\cdot\{\delta T_o\} + \sum_{m=1}^{M} Z_m\cdot\delta T_m^*\,. \tag{33}$$

Relationship (31) is subtracted from definition (32) to obtain a formula of propagation of temperature measurement error and the error of determination of the initial temperature onto a temperature distribution in the whole area. The matrix STAB is called the stability matrix. The inverse problem is stable when spectral radius r of the STAB matrix satisfies the condition $r<1$.

4. Numerical calculations

For the purpose of numerical calculations, the temperature distribution at points (ξ^*, t^*) has been assumed as equal to the solution of a simple problem under the conditions

$$T(\xi,0) = T_o = 1, \qquad \frac{\partial T}{\partial \xi}\bigg|_{\xi=0} = 0, \qquad \frac{\partial T}{\partial \xi}\bigg|_{\xi=1} = Bi(T-T_f), \qquad Bi = \frac{\alpha\cdot l}{\lambda}\,,$$

where Bi is Biot's number (3rd boundary condition) and T_f is the temperature of the liquid that surrounds the layer. If $Bi\to\infty$ the surface $\xi=1$ is subject to the boundary condition of the first type. The following values have been assumed for the calculations: the mesh in ξ-direction is divided into 30 parts, time step $\Delta\tau = 0.5$ s, the layer thickness $l = 58$ mm, the thermal conductivity $\lambda = 27.2$ W/mK, and the heat transfer coefficient is $\alpha = 20\,000$ W/mK, The temperature measurement points are located 2 mm from the edge ($\xi^* = 0.9655$).

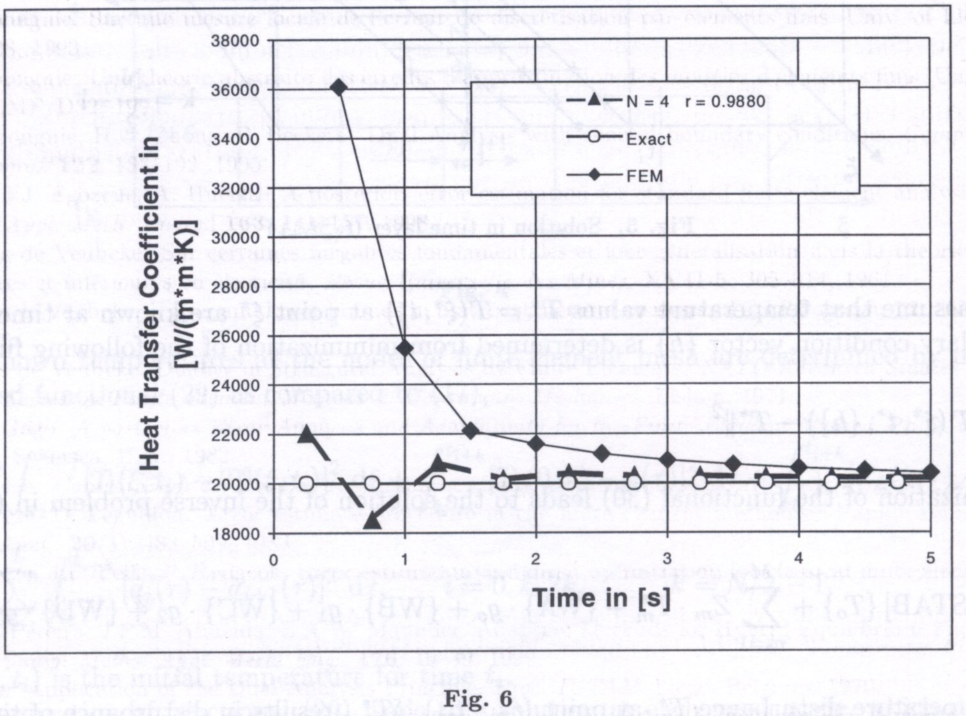


Fig. 6

Figures 6, 7 and 8 show the pattern of heat transfer coefficient determined from the solution of the inverse problem using the boundary condition of the first type (4) for linear approximation ($N = 4$).

The results in Figs. 6, 7 and 8 are obtained wit linear ($N = 4$) square ($N = 6$), and cubic ($N = 8$) approximations in time of function $T(\xi, t)$. Stability of the method is maintained. Values of spectral radii as functions of time step are shown in Fig. 9.

Tables 1–3 show the properties of the FEM with new basic functions in comparison with FEM including classical ones.

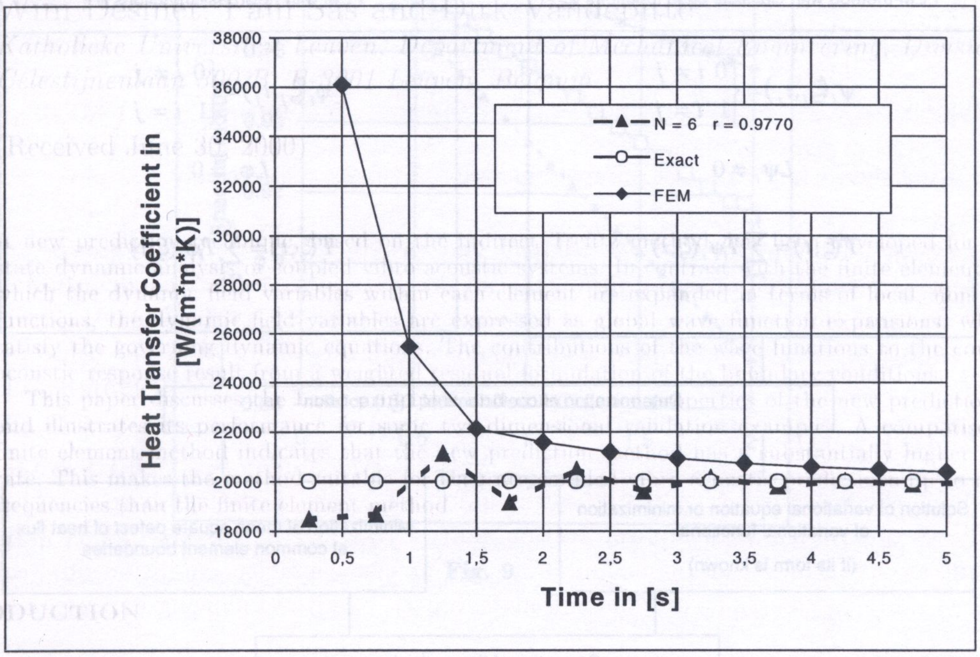


Fig. 7

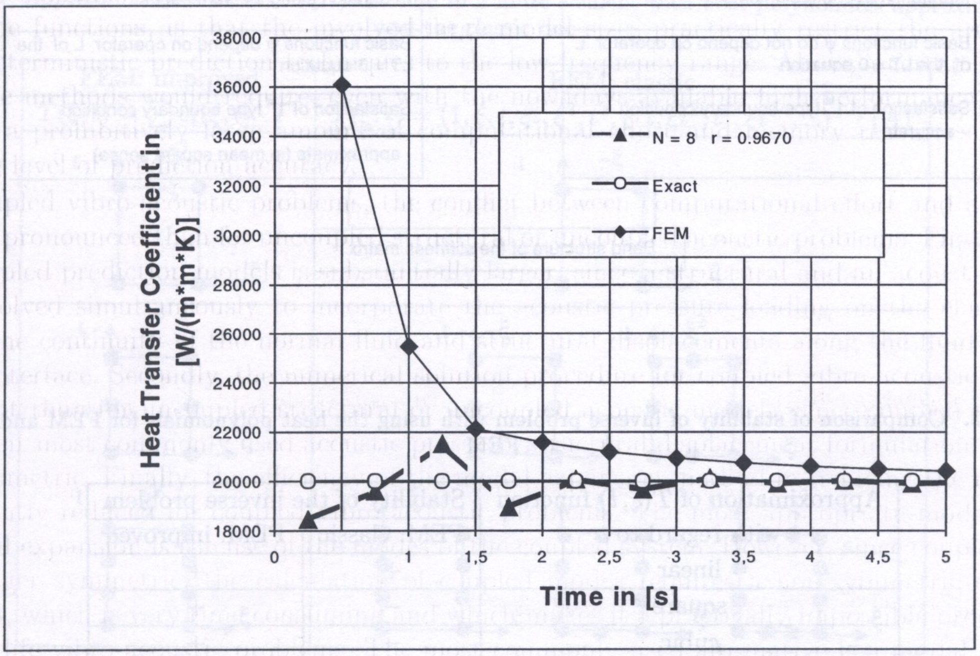


Fig. 8

Table 1. Comparson of FEM with classical and new basic functions

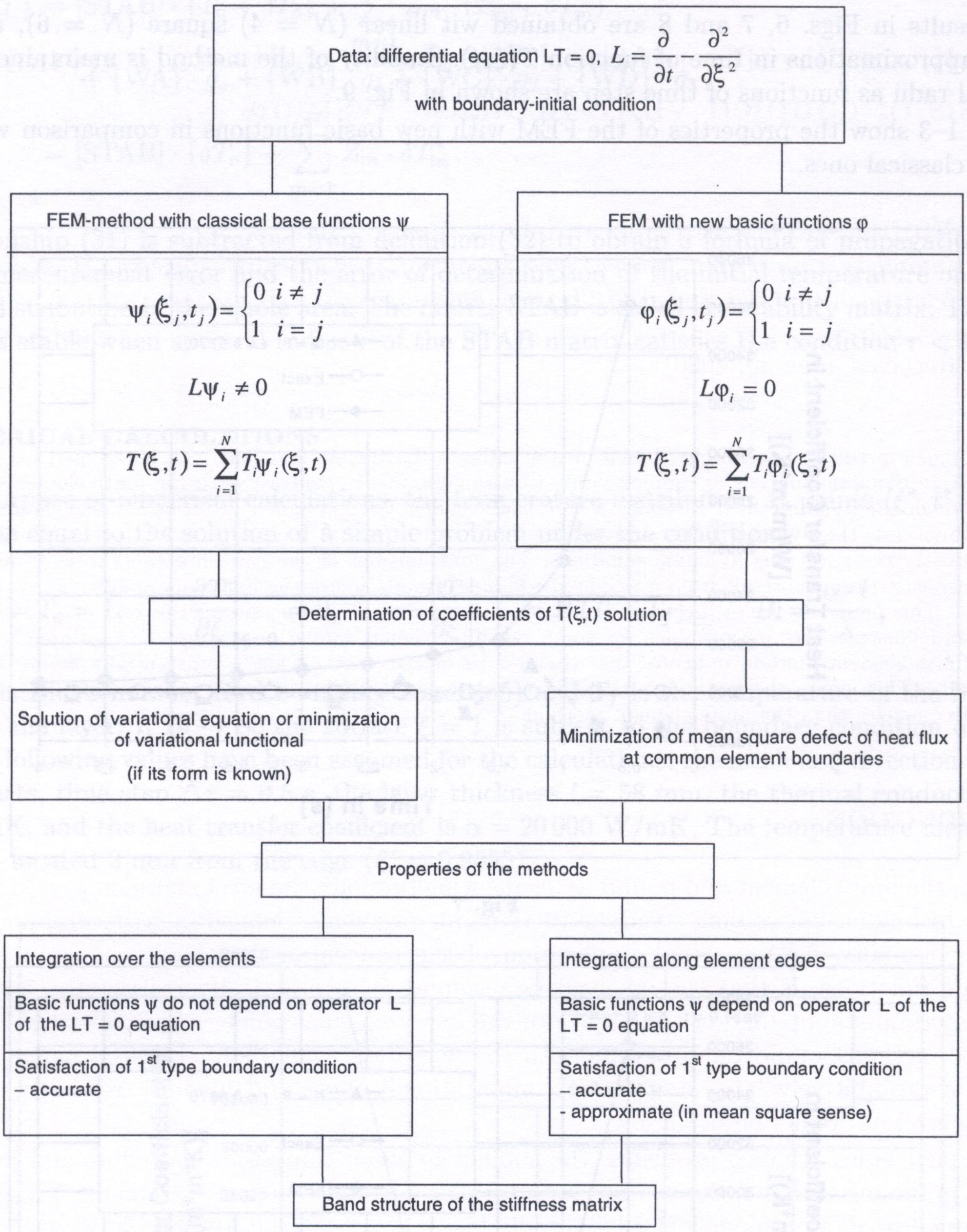


Table 2. Comparison of stability of inverse problem with using the heat polynomials for FEM and classic FEM

Approximation of $T(\xi, t)$ function with regard to t	Stability of the inverse problem	
	FEM: classic	FEM: improved
linear	+	+
square	−	+
cubic	−	+

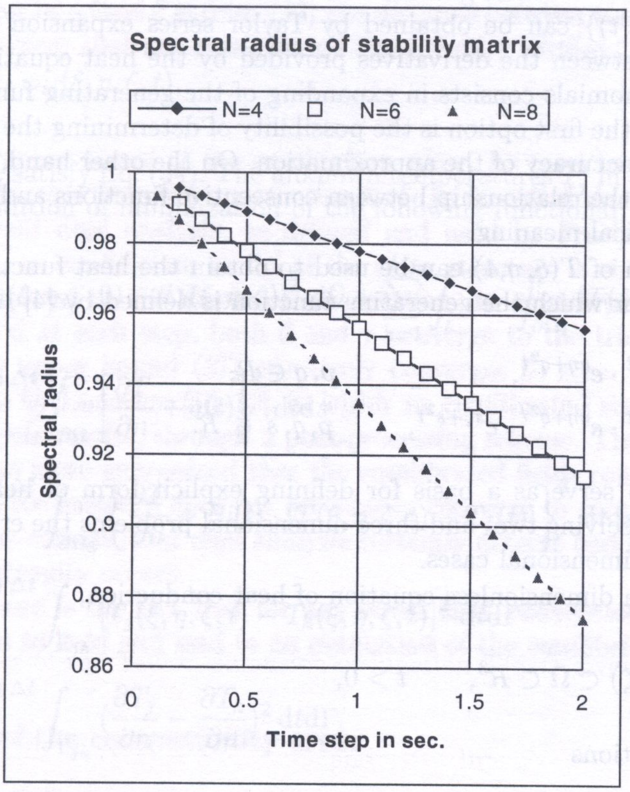
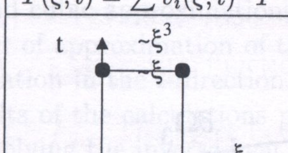
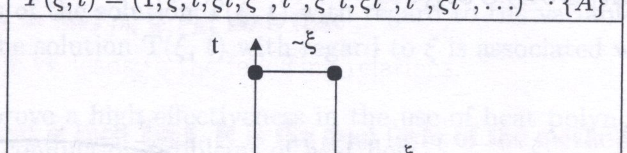
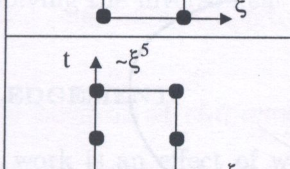
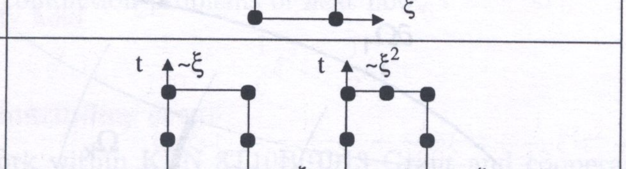
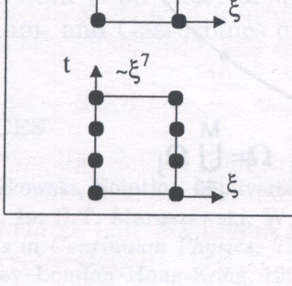
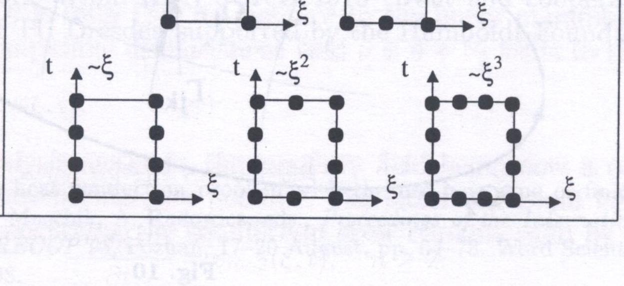


Fig. 9

Table 3. Approximation order of $T(\xi,t)$ solution in a finite element with heat polynomials approximation and the classic one

FEM: improved $T(\xi,t)=\sum v_i(\xi,t)\cdot T^i$	FEM: classic $T(\xi,t)=(1,\xi,t,\xi t,\xi^2,t^2,\xi^2 t,\xi t^2,t^3,\xi t^3,\ldots)^T\cdot\{A\}$
	
	
	

5. Generalization of the use of heat polynomials

Heat polynomials $\{\nu_n(\xi,t)\}$ can be obtained by Taylor series expansion of function $T(\xi,t)$ into using the relationship between the derivatives provided by the heat equation (8). Another way of obtaining the heat polynomials consists in expanding of the generating function (12) into a power series. The advantage of the first option is the possibility of determining the remainder of the Taylor series that indicates the accuracy of the approximation. On the other hand, the generating function enables the definition of the relationship between consecutive functions and their derivatives, which is of considerable numerical meaning.

The Taylor expansion of $T(\xi,\eta,t)$ can be used to obtain the heat functions for two- and three-dimensional problems, for which the generating function is defined by [4]

$$W(\xi,\eta,t) = e^{p\xi+p^2t}\cdot e^{q\eta+q^2t}, \qquad p,q\in R, \tag{34}$$

$$W(\xi,\eta,\zeta,t) = e^{p\xi+p^2t}\cdot e^{q\eta+q^2t}\cdot e^{s\zeta+s^2t}, \qquad p,q,s\in R. \tag{35}$$

Functions (34) and (35) serve as a basis for defining explicit form of heat polynomials in other coordinate systems. For solving two- and three-dimensional problems the error functional (29) must be generalized to multidimensional cases.

Let us consider now a dimensionless equation of heat conduction,

$$\frac{\partial T}{\partial t} = \Delta T, \qquad (\xi,\eta,\zeta)\subset\Omega\subset R^3, \qquad t>0,$$

with the following conditions

- initial conditions

$$T(\xi,\eta,\zeta,0) = T_o(\xi,\eta,\zeta),$$

- boundary conditions, Fig. 10 ,

$$T(\xi,\eta,\zeta,t)|_{\partial\Omega_1} = f(t), \qquad -\left.\frac{\partial T}{\partial n}\right|_{\partial\Omega_2} = q(t), \qquad -\left.\frac{\partial T}{\partial n}\right|_{\partial\Omega_3} = Bi(t)\left[T(\xi,\eta,\zeta,t)|_{\partial\Omega_3} - T_f(t)\right],$$

$$\partial\Omega = \partial\Omega_1\cup\partial\Omega_2\cup\partial\Omega_3 .$$

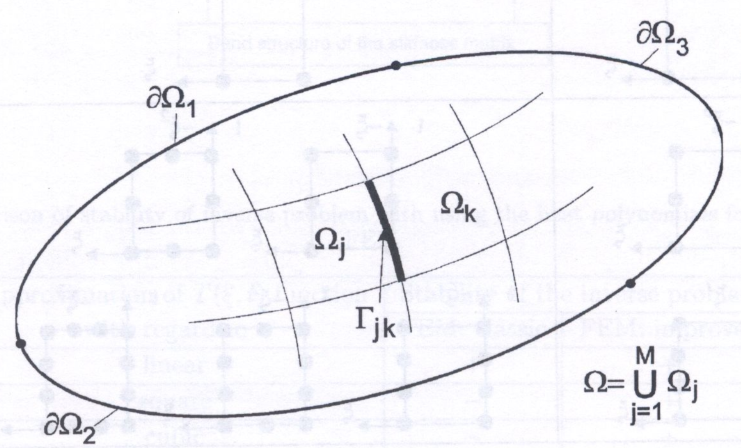


Fig. 10

The domain Ω, Fig. 10, is divided into finite elements Ω_j, then solution of the heat conduction equation in the element Ω_j may be presented in the following form,

$$T_j(\xi,\eta,\zeta,t) = \sum_{i=1}^{N} T_j^i \cdot \varphi_{ij}(\xi,\eta,\zeta,t), \tag{36}$$

where base functions φ_{ij} satisfy Eq. (34). The unknown temperatures T_j^i in i-nodes of the Ω_j element are sought from the condition of minimization of the following functional,

$$\begin{aligned} J(\{T\}) = {} & \sum_{j=1}^{M_0} \int_{\Omega_j} [T_j(\xi,\eta,\zeta,0) - T_o(\xi,\eta,\zeta)]^2 \, d\Omega + \sum_{k=1}^{M_1} \int_t^{t+\Delta t} \int_{\partial\Omega_1} [T(\xi,\eta,\zeta,t) - f(t)]^2 \, dtds \\ & + \sum_{k=1}^{M_2} \int_t^{t+\Delta t} \int_{\partial\Omega} \left[\frac{\partial T}{\partial n} + q(t)\right]^2 dtds \\ & + \sum_{k=1}^{M3} \int_t^{t+\Delta t} \int_{\partial\Omega_3} \left\{\frac{\partial T}{\partial n} + Bi(t) \cdot [T(\xi,\eta,\zeta,t) - T_f(t)]\right\}^2 dtds \\ & + \sum \int_t^{t+\Delta t} \int_{\Gamma_{jk}} [T_j(\xi,\eta,\zeta,t) - T_k(\xi,\eta,\zeta,t)]^2 \, dtd\Gamma \\ & + \sum \int_t^{t+\Delta t} \int_{\Gamma_{jk}} \left(\frac{\partial T_j}{\partial n} - \frac{\partial T_k}{\partial n}\right)^2 dtd\Gamma, \qquad \Omega = \bigcup_{j=1}^{M} \Omega_j \, . \end{aligned} \tag{37}$$

Minimization of functional (37) leads to a system of equations that includes the temperatures T_j^i (36) at the nodes of the finite elements. The functional (37) includes integration over an element surface with regard to initial temperature, while in the case of boundary conditions and heat flux the integration is made at the element boundaries.

6. Conclusions

The introduction of functional heat flux defect at common boundaries of elements in the process of solution of the problem of heat flow provides stable solution of the inverse problem, in both cases of square and cubic approximations of the solution $T(\xi, t)$ with regard to the variable t.

The order of approximation of the solution $T(\xi, t)$ with regard to ξ is associated with the order of approximation in the t-direction.

The results of the calculations prove a high effectiveness in the use of heat polynomials for the purpose of solving the inverse heat conduction problems of heat flow.

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