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Comparison of two types of Trefftz method for the solution of inhomogeneous elliptic problems

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The solution of inhomogeneous elliptic problems by the Trefftz method has become increasingly more popular during the last decade [1–3]. One method of solution uses the fundamental solutions as trial functions and the inhomogeneous part is expressed by radial basis functions (RBFs).

The purpose of this paper is to solve several boundary value problems that have exact solutions. Two error criteria are used for comparison of the exact solutions and the approximated solutions. The first is the mean least square global error. The second has a local character, as it measures the absolute maximal error.

1. Description of the first method

Consider a general differential equation,

$$\nabla^2 u = f(x,y) \quad \text{in} \quad \Omega, \quad \text{and} \quad Bu = g(x,y) \quad \text{on} \quad \partial\Omega, \tag{1}$$

where ∇^2 is the Laplace differential operator, and B is an operator imposed as boundary conditions, such as Dirichlet, Neumann, and Robin.

Let $\{P_i = (x_i, y_i)\}_{i=1}^{N}$ denote the set of N collocations points in Ω, of which $\{(x_i, y_i)\}_{i=1}^{Nl}$, are interior points and $\{(x_i, y_i)\}_{i=Nl+1}^{N}$ are boundary points.

The right-hand side function f is approximated by RBFs as,

$$f_N(x,y) = \sum_{j=1}^{N} a_j \varphi(r_j) + \sum_{k=1}^{l} b_k p_k(x,y), \tag{2}$$

where $\varphi(r_j) = \varphi\left(\sqrt{(x-x_j)^2 + (y-y_j)^2}\right) : R^d \to R^+$ is a RBF, $\{p_k\}_{k=1}^{l}$ is the complete basis for d-variate polynomials of degree $\le m-1$, and C_{m+d-1}^{d} is the dimension of p_{m-1}. The coefficients $\{a_j\}$, $\{b_k\}$ can be found by solving the system,

$$\begin{gathered}\sum_{j=1}^{N} a_j \varphi(r_{ji}) + \sum_{k=1}^{l} b_k p_k(x_i, y_i) = f(x_i, y_i), \quad 1 \le i \le N, \\ \sum_{j=1}^{N} a_j p_k(x_j, y_j) = 0, \quad 1 \le k \le l,\end{gathered} \tag{3}$$

where $r_{ji} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, and $\{x_i, y_i\}_{i=1}^{N}$ are the collocation points on $\Omega \cup \partial\Omega$.

The approximate particular solutions u_p can be obtained using coefficients $\{a_j\}$ and $\{b_k\}$,

$$u_p(x,y) = \sum_{j=1}^{N} a_j \Phi_j(x,y) + \sum_{k=1}^{N} b_k \Psi_k(x,y), \tag{4}$$

where

$$\nabla^2 \Phi_j(x,y) = \varphi_j(x,y), \quad \text{for} \quad j = 1,\ldots,N, \tag{5}$$

$$\nabla^2 \Psi_k(x,y) = p_k(x,y), \quad \text{for} \quad k = 1,\ldots,l. \tag{6}$$

The solution of differential Eq. (1) can be given as,

$$u = u_p + v, \tag{7}$$

where v is the solution of the boundary value problem in the form:

$$\nabla^2 v = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad Bv = g(x,y) - Bu_p \quad \text{on} \quad \partial\Omega. \tag{8}$$

The method of fundamental solutions is used to solve problem (6), meaning that:

$$v(x,y) = \sum_{k=1}^{NC} c_k \ln\left[(x-x_k)^2 + (y-y_k)^2\right] \tag{9}$$

Enforcement of the boundary conditions yields,

$$\sum_{k=1}^{NC} c_k B \ln\left[(x_i-x_k)^2 + (y_i-y_k)^2\right] = g(x_i,y_i) - Bu_p(x_i,y_i), \tag{10}$$

where NC is the number of collocation points.

The inhomogeneous boundary problem (1) has been solved by numerical implementation of the solution given by Eq. (7). Function f is approximated as stated in Eq. (2). The radial basis functions φ that are used are defined as follows:

Case I

$$\varphi(r_j) = 1 + r_j. \tag{11}$$

Case II

$$\varphi(r_j) = \begin{cases} 0 & \text{for} \quad r_j = 0, \\ r_j^2 \ln r_j & \text{for} \quad r_j \neq 0. \end{cases} \tag{12}$$

Case III

$$\varphi(r_j) = \sqrt{r_j^2 + C^2}. \tag{13}$$

Case IV

$$\varphi(r_j) = r_j^2 + r_j^3. \tag{14}$$

Case V

$$\varphi_j = \begin{cases} \left(1 - \dfrac{r_j}{a}\right)^4 \left(1 + \dfrac{4r_j}{a}\right) & \text{for} \quad 0 \le r_j \le a, \\ 0 & \text{for} \quad r_j > a, \end{cases} \tag{15}$$

where C is a parameter and $r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}$ for j=1,2,...,N.

For the problems (5) the solutions (respectively to RBF defined in Eqs. (11)–(15)) are presented by Eqs. (16)–(20).

Case I

$$\Phi_j = \frac{r_j^2}{4} + \frac{r_j^3}{9}. \tag{16}$$

Case II

$$\Phi_j = \frac{r_j^4 \ln r_j}{16} - \frac{r_j^4}{32}. \tag{17}$$

Case III

$$\Phi_j = -\frac{c^2 \ln\left[c\left(r_j^2+c^2\right)^{\frac{1}{2}} + c^2\right]}{3} + \frac{\left[\left(r_j^2+4c^2\right)\left(r_j^2+c^2\right)^{\frac{1}{2}}\right]}{9}. \tag{18}$$

Case IV

$$\Phi_j = \frac{r_j^4}{16} + \frac{r_j^5}{25}. \tag{19}$$

Case V

$$\Phi_j = \begin{cases} \dfrac{r_j^2}{4} - \dfrac{5r_j^4}{8a^2} + \dfrac{4r_j^5}{5a^3} - \dfrac{5r_j^6}{12a^4} + \dfrac{4r_j^7}{49a^5} & \text{for} \quad 0 \le r_j \le a, \\ \dfrac{529a^2}{5880} + \dfrac{a^2}{14}\ln\left(\dfrac{r}{a}\right) & \text{for} \quad r_j > a, \end{cases} \tag{20}$$

where C is a parameter and $r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}$ for j=1,2,...,N.

2. Description of the second method

The solution of the Poisson equation,

$$\nabla^2 u = \Delta u = f, \tag{21}$$

may be rewritten using the inverse Laplace operator (see Ref. [4]),

$$u = \Delta^{-1} f + H, \tag{22}$$

where $\Delta^{-1} f$ is a particular solution and harmonic function H is the general solution of the Poisson equation related to the boundary conditions.

The approximation of the source function, the calculation of the inverse Laplace operator and interpolation of the harmonic function,

$$H = u - \Delta^{-1} f, \tag{23}$$

are used to solve the Poisson problem with its boundary conditions.

For 2D problems the harmonic functions are in the form:

$$H(x,y) = A_0 + \sum_{n=1}^{\infty} \left[A_n F_n(x,y) + B_n G_n(x,y)\right], \tag{24}$$

where functions F_n, G_n are obtained using the formulae:

$$e^{x+iy} = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!} = \sum_{n=0}^{\infty} \left[F_n(x,y) + \mathrm{i}G_n(x,y)\right]. \tag{25}$$

The infinite series is truncated to a finite number of terms for numerical calculations.

Function $f(x,y)$ is expanded in Taylor series in the domain Ω:

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} x^k y^{n-k}. \tag{26}$$

Therefore, the calculation of $\Delta^{-1} f$ is equivalent to obtaining $\Delta^{-1}(x^k y^{n-k})$:

$$\Delta^{-1} f(x,y) = \Delta^{-1} \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} x^k y^{n-k} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} \Delta^{-1} \left(x^k y^{n-k} \right). \tag{27}$$

The expression given above may be calculated by two methods, namely the recurrence scheme and the transfer of the power series into a series of polyharmonic functions.

2.1. Recurrence scheme

Using the notation,

$$Q_{ij} = x^i y^j, \tag{28}$$

the following formulae are found:

$$\Delta Q_{ij} = 0, \qquad i,j = \{0,1\}, \tag{29}$$

$$\Delta Q_{ij} = j(j-1) x^i y^{j-2} = j(j-1) Q_{i,j-2}, \qquad i = \{0,1\}, \quad j > 1, \tag{30}$$

$$\Delta Q_{ij} = i(i-1) x^{i-2} y^j = i(i-1) Q_{i-2,j}, \qquad i > 1, \quad j = \{0,1\}, \tag{31}$$

$$\begin{aligned}\Delta Q_{ij} &= \Delta\left(x^i y^j\right) = i(i-1) x^{i-2} y^j + j(j-1) x^i y^{j-2} \\ &= i(i-1) Q_{i-2,j} + j(j-1) Q_{i,j-2}, \qquad i,j > 1.\end{aligned} \tag{32}$$

The following recurrence scheme is obtained after applying the inverse Laplace operator:

$$\Delta^{-1} Q_{ij} = \frac{1}{(i+2)(i+1)} Q_{i+2,j}, \qquad i \geq 0, \quad j = \{0,1\}, \tag{33}$$

$$\Delta^{-1} Q_{ij} = \frac{1}{(i+2)(i+1)} Q_{i+2,j} - \frac{j(j-1)}{(i+2)(i+1)} \Delta^{-1} Q_{i+2,j-2}, \qquad i \geq 0, \quad j \geq 2. \tag{34}$$

2.2. Polyharmonic functions series

Using the transformation

$$x = \frac{z+\overline{z}}{2}, \qquad y = \frac{z-\overline{z}}{2\mathrm{i}}, \tag{35}$$

the polyharmonic functions series is obtained from power series:

$$f(z,\overline{z}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{nk} z^k \overline{z}^{n-k}, \qquad c_{nk} = \overline{c}_{n,n-k}, \tag{36}$$

$$\Delta^{-1} f(z,\overline{z}) = \Delta^{-1}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{nk} z^k \overline{z}^{n-k}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{nk} \Delta^{-1}\left(z^k \overline{z}^{n-k}\right). \tag{37}$$

The calculation of the inverse Laplace operator yields:

$$\Delta\left(z^{i+1}\overline{z}^{j+1}\right) = 4(i+1)(j+1) z^i \overline{z}^j, \tag{38}$$

$$\Delta^{-1}\left(z^i \overline{z}^j\right) = \frac{z^{i+1}\overline{z}^{j+1}}{4(i+1)(j+1)}. \tag{39}$$

Therefore:

$$\begin{aligned}\Delta^{-1} f(z,\overline{z}) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_{nk}}{4(k+1)(n-k+1)} z^{k+1}\overline{z}^{n-k+1} \\ &= r^2 \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_{nk}}{4(k+1)(n-k+1)} z^{k}\overline{z}^{n-k},\end{aligned} \tag{40}$$

where $r^2 = z\overline{z}$.

2.3. Numerical implementation

The domain Ω is a unit square, divided in a mesh with N nodes. N is equal to the number of basis functions:

$$f_{nk}(x,y) = \mathrm{Re}\left(z^k \overline{z}^{n-k}\right), \tag{41}$$

$$g_{nk}(x,y) = \mathrm{Im}\left(z^k \overline{z}^{n-k}\right). \tag{42}$$

The basis functions are polynomial of degree n. The number of such polynomials is given by:

$$S_n = \frac{(n+1)(n+2)}{2}. \tag{43}$$

The power series is an expansion of a function in a Fourier series. The truncation of the series to S_n terms causes error of order $O(h^{n+1})$. However the condition

$$S_n = N, \tag{44}$$

is not always fulfilled.

If it is not fulfilled, the next terms of the series are taken into account, which does not decrease the error. The density of the mesh in both x and y direction is the same. Thus N is the square of a whole number, and the condition is not fulfilled.

The transformation of function $f(x,y)$ from a complex form into a real form is defined by:

$$f(z,\overline{z}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{nk} z^k \overline{z}^{n-k}, \qquad c_{nk} = \overline{c}_{n,n-k}, \tag{45}$$

$$c_{nk} = \frac{a_{nk} - ib_{nk}}{2}, \qquad c_{n,n-k} = \frac{a_{n,n-k} - ib_{n,n-k}}{2} = \frac{a_{nk} + ib_{nk}}{2}, \tag{46}$$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \left[a_{nk}\mathrm{Re}\left(z^k\overline{z}^{n-k}\right) + b_{nk}\mathrm{Im}\left(z^k\overline{z}^{n-k}\right)\right] = \sum_{k=1}^{N} w_k p_k(x,y), \tag{47}$$

where functions $p_k(x,y)$ are of the form $\mathrm{Re}\left(z^k\overline{z}^{n-k}\right)$, $\mathrm{Im}\left(z^k\overline{z}^{n-k}\right)$. Functions f_{nk} and g_{nk} satisfy Eq. (48).

$$\Delta^M f_{nk} = \Delta^M g_{nk} = 0, \qquad M = k+1\,. \tag{48}$$

Therefore, they are polyharmonic functions. Functions f_{nk} and g_{nk} are set up in sequence and are denoted by p_k. If the polyharmonic functions are chosen the Eq. (49) can be written for each node.

$$f_i = f(x_i, y_i) = \sum_{k=1}^{N} w_k p_k(x_i, y_i) = \sum_{k=1}^{N} w_k p_{ki}\,. \tag{49}$$

The system of equations given above can be written in matrix form,

$$\mathbf{f} = \mathbf{A}\mathbf{w}, \tag{50}$$

where $\mathbf{f} = \{f_i\}_{i=1,\dots,N}$, $\mathbf{A} = \{p_{ki}\}_{k=1,\dots,N\times i=1,\dots,N}$, $\mathbf{w} = \{w_k\}_{k=1,\dots,N}$. As matrix $\mathbf{A}$ is ill-conditioned, the solution in the form of Eq. (51),

$$\mathbf{w} = \mathbf{A}^{-1}\mathbf{q}, \tag{51}$$

is avoided. The system of equations is solved by the least-squares approximation,

$$\mathbf{w} = \mathbf{A}^{I}\mathbf{q}, \tag{52}$$

where $\mathbf{A}^I$ is the pseudoinverse matrix. The SVD (Singular Value Decomposition) algorithm has been used to obtain the pseudoinverse matrix. The order of the pseudoinverse matrix, $\mathbf{M}$, is defined below for each example. Next, the solution of the Poisson equation is found:

$$\Delta^{-1} f(x,y) = \sum_{k=1}^{N} w_k \Delta^{-1} p_k(x,y)\,. \tag{53}$$

The value of the right-hand side of Eq. (22) is calculated on the boundary. The function is an harmonic function (i.e. it fulfils the Laplace equation) and satisfies the boundary condition.

3. The numerical calculations

The exact solution is known for each example considered. Two error criteria are used to confirm the accuracy of the solutions calculated numerically for the inhomogeneous boundary problem (1). The criteria used are:

$\varepsilon_{\max}$ – the global maximal error

$$\varepsilon_{\max} = \max \frac{|u_i^e - u_i^a|}{|u_{\max}^e|}, \tag{54}$$

ε_{ms} – the mean square error:

$$\varepsilon_{ms} = \frac{1}{|u^a_{max}|}\sqrt{\frac{1}{NP}\sum_{i=1}^{NP}(u^e_i - u^a_i)^2}. \tag{55}$$

A distance between source and collocation points is introduced (see Fig. 1), to avoid the singularity of the fundamental solution function. The influence of this distance, s, on the calculated results is investigated.

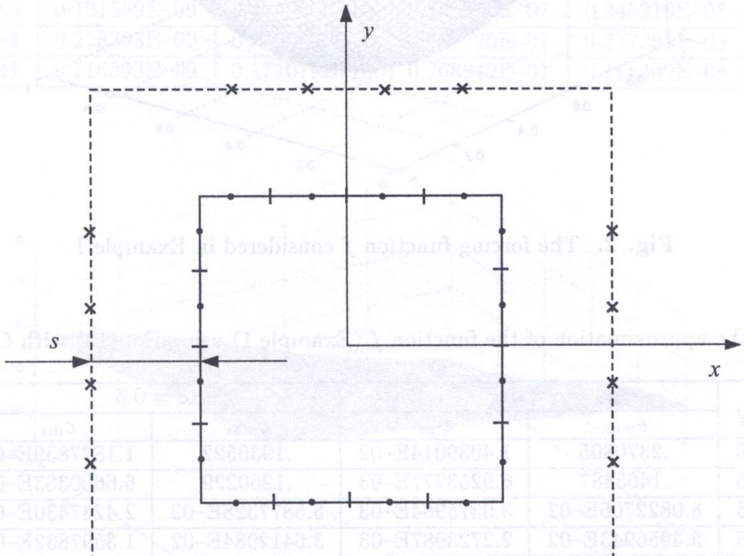

Fig. 1. The domain of the problems considered, with source points (×), collocation points (•) and distance s indicated

All test solutions on problem (1) presented below are implemented on the domain $\Omega = [0,1] \times [0,1]$. The number of collocation points is $NC4 = 4 \cdot NC$.

Example 1

The forcing term and the boundary conditions considered in the first set of tests are:

$$f(x,y) = -2\left(x + y - x^2 - y^2\right), \quad \text{and} \quad u(x,y) = 0 \quad \text{on} \quad \partial\Omega.$$

This function f is represented in Fig. 2. The exact solution of this problem is

$$u(x,y) = \left(x - x^2\right)\left(y - y^2\right).$$

Table 1 shows that there is a very good approximation of the function f by the chosen radial basis functions. The errors indicated in Table 2 confirm the accuracy of solutions obtained by the method of fundamental solutions.

The results in Table 3 show that good approximations are obtained for function f and for the solution of the problem using the second solution method.

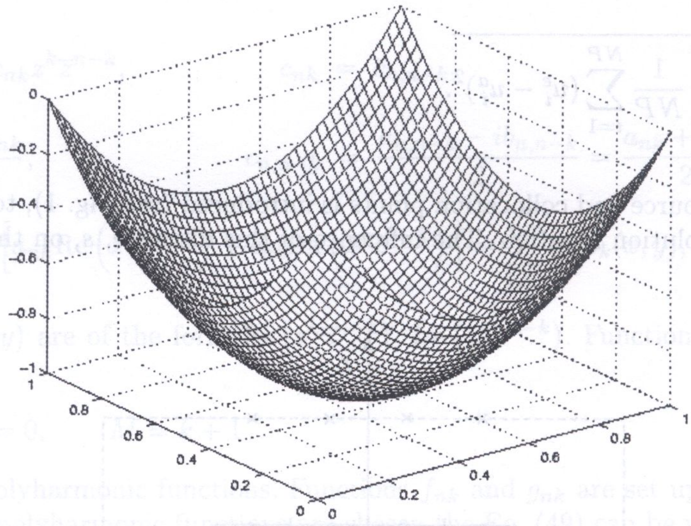

Fig. 2. The forcing function f considered in Example 1

Table 1. Error in the approximation of the function f (Example 1) using Eq. (13) with $C = 0.2$ and $S = 0.1$

N	$c = 0.2$		$c = 0.3$	
	ε_{max}	ε_{ms}	ε_{max}	ε_{ms}
9	.2370805	1.4039014E–02	.1930522	1.1877839E–02
16	.1465487	6.9253277E–03	.1250229	6.6600363E–03
25	8.0822706E–02	3.0375984E–03	5.5877328E–02	2.4247430E–03
36	5.4956943E–02	2.2723987E–03	3.6412984E–02	1.3597832E–03
49	3.6066055E–02	1.2205222E–03	2.0247996E–02	6.7211705E–04
64	2.5565803E–02	6.8597513E–04	1.4154106E–02	3.8922983E–04
81	1.8589854E–02	5.2528735E–04	1.3733625E–02	4.0787607E–04
100	1.2621224E–02	3.1476223E–04	1.9448578E–02	9.9884078E–04
121	1.1562243E–02	3.0307425E–04	.3016626	1.8310498E–02

Table 2. Error in the solution of the inhomogeneous boundary problem (Example 1) using Eq. (13) with $C = 0.2$ and $S = 0.01$

$NC4$	ε_{max}	ε_{ms}
40	.1608382	5.5307264E–05
44	.1292201	3.1719854E–05
48	.1046818	1.8679842E–05
52	8.5394941E–02	1.1243576E–05
56	7.0066795E–02	6.8975992E–06
60	5.7761610E–02	4.3014884E–06
64	4.7822256E–02	2.7213302E–06
68	3.9720606E–02	1.7441691E–06
72	3.3110030E–02	1.1319100E–06
76	2.7673844E–02	7.4239512E–07
80	2.3203671E–02	4.9194807E–07
84	1.9483345E–02	3.2965215E–07
88	1.6407317E–02	2.2292440E–07
92	1.3825138E–02	1.5184744E–07
96	1.1696047E–02	1.0508268E–07
100	9.8939491E–03	7.3039438E–08

Table 3. Error in the approximation of the function f and of the solution of the inhomogeneous boundary problem (Example 1) by the polyharmonic functions method

N	f		u	
	$\varepsilon_{\max}$	ε_s	$\varepsilon_{\max}$	ε_s
9	0.177636E–14	0.419004E–15	0.111022E–14	0.291419E–15
25	0.444089E–14	0.146664E–14	0.324740E–14	0.987639E–15
49	0.384683E–07	0.249855E–07	0.794045E–06	0.244638E–06
81	0.107188E–09	0.690977E–10	0.380602E–08	0.100325E–08
121	0.500239E–12	0.315880E–12	0.291419E–10	0.623190E–11
169	0.783526E–12	0.145587E–12	0.114406E–09	0.693674E–11
225	0.981989E–11	0.172815E–11	0.132993E–08	0.856409E–10
289	0.191589E–09	0.424966E–10	0.433373E–07	0.245210E–08
361	0.228393E–09	0.349009E–10	0.465720E–07	0.237231E–08
441	0.116393E–09	0.171079E–10	0.208942E–07	0.114409E–08

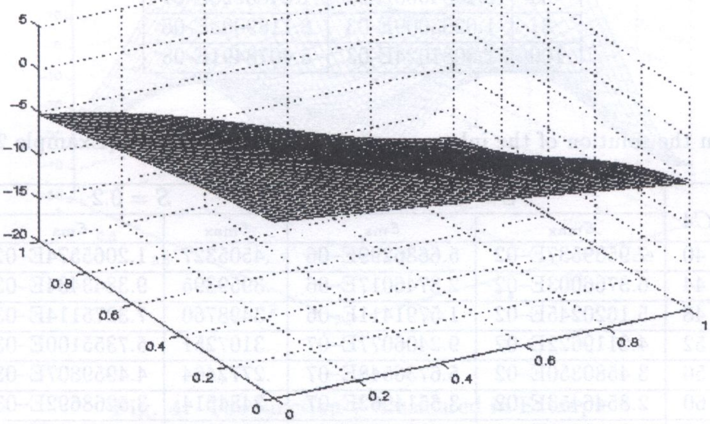

Fig. 3. The function f considered in Example 2

Example 2

The forcing term and the boundary conditions considered in the second set of tests are:

$$f(x,y) = -6x - 6y - \left[\frac{4}{\alpha^2} - 4\left(\frac{x-\beta}{\alpha^2}\right)^2 - 4\left(\frac{y-\beta}{\alpha^2}\right)^2\right]\exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right],$$

and

$$u(x=0) = -y^3 + \exp\left[-\left(\frac{\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right],$$

$$u(x=1) = -1 - y^3 + \exp\left[-\left(\frac{1-\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right],$$

$$u(y=0) = \frac{2\beta}{\alpha} + \exp\left[-\left(\frac{\beta}{\alpha}\right)^2 - \left(\frac{x-\beta}{\alpha}\right)^2\right],$$

$$u(y=1) = -3 - 2\frac{1-\beta}{\alpha}\exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right].$$

The exact solution of this is given by:

$$u(x,y) = -x^3 - y^3 + \exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right].$$

The calculations have been performed for $\alpha = 0.8, \beta = 1.0$.

The relatively good approximation obtained for function f using the radial basis functions in form (14) is shown in Table 4. The approximate solution is obtained with good accuracy, as well (see Table 5). The method based on polyharmonic functions gives correct results as well.

Table 4. Error in the approximation of function f (Example 2) using Eq. (14)

N	$\varepsilon_{\max}$	ε_{ms}
9	7.5146712E-02	9.6021872E-04
16	1.7227121E-02	2.3231516E-05
25	8.7877037E-03	5.3437820E-06
36	4.7276118E-03	1.1288827E-06
49	3.1483271E-03	4.1058800E-07
64	2.1704060E-03	1.5483226E-07
81	1.6283009E-03	8.1189093E-08
100	1.3087024E-03	5.4078491E-08

Table 5. Error in the solution of the inhomogeneous boundary problem (Example 2) using Eq. (14)

$NC4$	$S = 0.1$		$S = 0.2$	
	$\varepsilon_{\max}$	ε_{ms}	$\varepsilon_{\max}$	ε_{ms}
40	7.9539537E-02	5.6686299E-06	.4505327	1.2065574E-02
44	6.3766003E-02	2.8746017E-06	.3959395	9.3843434E-03
48	5.1620245E-02	1.5791411E-06	.3498760	7.3275114E-03
52	4.2119622E-02	9.2406077E-07	.3107251	5.7355100E-03
56	3.4580350E-02	5.6736548E-07	.2772464	4.4959807E-03
60	2.8546453E-02	3.6514962E-07	.2484514	3.5268692E-03
64	2.3673892E-02	2.4660204E-07	.2235594	2.7675193E-03
68	1.9696236E-02	1.7170593E-07	.2019358	2.1715669E-03
72	1.6443253E-02	1.2598420E-07	.1830722	1.7035156E-03
76	1.3768792E-02	9.7632373E-08	.1665487	1.3359719E-03
80	1.1592865E-02	8.2259589E-08	.1520147	1.0473205E-03
84	9.7477436E-03	6.8561199E-08	.1391798	8.2065666E-04
88	8.2062483E-03	5.7730514E-08	.1278061	6.4283743E-04
92	6.9289207E-03	5.2247330E-08	.1176900	5.0338404E-04
96	5.8757067E-03	5.0956356E-08	.1086649	3.9408455E-04
100	4.9625635E-03	4.2635062E-08	.1005799	3.0844563E-04

Table 6. Error in the approximation of function f and of the solution of the inhomogeneous boundary problem (Example 2) by the polyharmonic functions method

N	f		u	
	$\varepsilon_{\max}$	ε_s	$\varepsilon_{\max}$	ε_s
9	0.413358E-01	0.141564E-01	0.734870E-01	0.258774E-01
25	0.573608E-03	0.240479E-03	0.569809E-02	0.155127E-02
49	0.290117E-04	0.674678E-05	0.615327E-03	0.934286E-04
81	0.122508E-05	0.226701E-06	0.462798E-04	0.465700E-05
121	0.356011E-07	0.698316E-08	0.210827E-05	0.174717E-06
169	0.870991E-09	0.197388E-09	0.764353E-07	0.580927E-08
225	0.146693E-10	0.598147E-11	0.170468E-08	0.210702E-09
289	0.113862E-09	0.262872E-10	0.200133E-07	0.122892E-08
361	0.136557E-09	0.202417E-10	0.220465E-07	0.112510E-08
441	0.747972E-10	0.904118E-11	0.160978E-07	0.717009E-09

Example 3

The forcing function (represented in Fig. 4) and boundary conditions used in the third set of tests are:

$$f(x,y) = \sin(px)\sin(qy) \quad \text{and} \quad u(x,y) = 0 \quad \text{on} \quad \partial\Omega .$$

The exact solution of the problem is given by $u(x,y) = -\dfrac{1}{p^2+q^2}\sin(px)\sin(qy)$. The calculations have been made with $p = 4\pi$ and $q = 4\pi$.

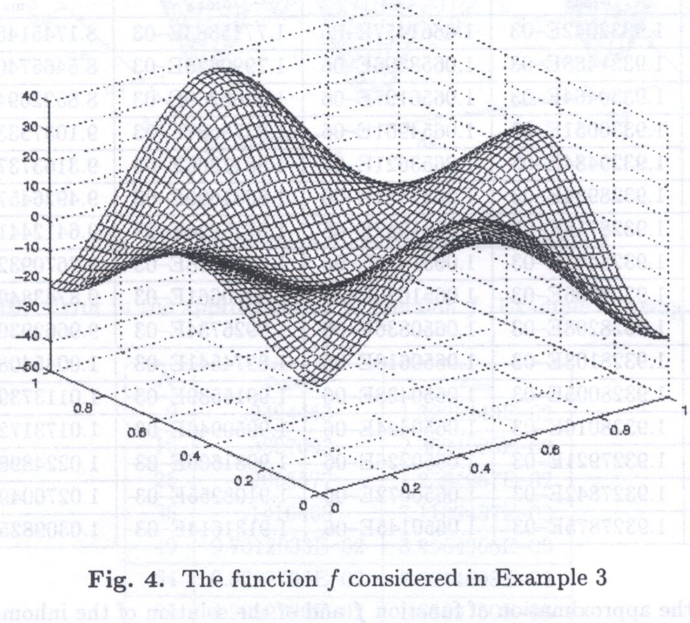

Fig. 4. The function f considered in Example 3

The method based on RBFs yields good approximations for the function f (Table 7) and for the solution (Table 8). Unfortunately, the second method considered involves quite large errors (for a low number of terms in the series). The accuracy of the method improves when the number of terms in the series is increased. Errors equivalent to those obtained with the first method are obtained with much higher computational effort (Table 9).

Table 7. Error in the approximation of function f (Example 3) using Eq. (12)

N	$\varepsilon_{\max}$	$\varepsilon_{\mathrm{ms}}$
9	3.8297747E–03	5.5551723E–06
16	3.8297747E–03	5.5551723E–06
25	3.6047916E–03	3.3331335E–06
36	3.8796824E–03	3.4999107E–06
49	3.8297747E–03	5.5551723E–06
64	3.8645109E–03	3.6241070E–06
81	2.9166634E–03	3.3797976E–06
100	3.5523642E–03	2.8986317E–06

Example 4

The forcing function and the boundary conditions used in the fourth set of tests are:

$$f(x,y) = -\frac{751\pi^2}{144}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4} + \frac{7\pi^2}{12}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4} + \frac{15\pi^2}{8}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4},$$

which is shown in Fig. 5 and

$$u(x,y) = 0 \quad \text{on} \quad \partial\Omega.$$

Table 8. Error in the solution of the inhomogeneous boundary problem (Example 3) using Eq. (12)

$NC4$	$S = 0.1$		$S = 0.01$	
	$\varepsilon_{\max}$	$\varepsilon_{\mathrm{ms}}$	$\varepsilon_{\max}$	$\varepsilon_{\mathrm{ms}}$
40	1.9332042E–03	1.0661457E–06	1.7715383E–03	8.1745145E–07
44	1.9331488E–03	1.0658896E–06	1.7990620E–03	8.5465740E–07
48	1.9330464E–03	1.0656197E–06	1.8206269E–03	8.8522694E–07
52	1.9330051E–03	1.0654901E–06	1.8378687E–03	9.1057933E–07
56	1.9329484E–03	1.0653621E–06	1.8517866E–03	9.3163737E–07
60	1.9328984E–03	1.0652531E–06	1.8631699E–03	9.4926457E–07
64	1.9328718E–03	1.0651947E–06	1.8725926E–03	9.6412441E–07
68	1.9328465E–03	1.0651412E–06	1.8804518E–03	9.7670932E–07
72	1.9328296E–03	1.0651030E–06	1.8870661E–03	9.8743840E–07
76	1.9328208E–03	1.0650836E–06	1.8926734E–03	9.9662930E–07
80	1.9328108E–03	1.0650618E–06	1.8974541E–03	1.0045408E–06
84	1.9328005E–03	1.0650438E–06	1.9015539E–03	1.0113739E–06
88	1.9328016E–03	1.0650424E–06	1.9050946E–03	1.0173172E–06
92	1.9327921E–03	1.0650226E–06	1.9081606E–03	1.0224898E–06
96	1.9327842E–03	1.0650072E–06	1.9108255E–03	1.0270049E–06
100	1.9327875E–03	1.0650145E–06	1.9131614E–03	1.0309825E–06

Table 9. Error in the approximation of function f and of the solution of the inhomogeneous boundary problem (Example 3) by the polyharmonic functions method

N	f		u	
	$\varepsilon_{\max}$	ε_s	$\varepsilon_{\max}$	ε_s
9	0.100000E+01	0.497009E+00	0.100000E+01	0.497009E+00
25	0.100000E+01	0.497009E+00	0.100000E+01	0.497009E+00
49	0.118764E+01	0.458319E+00	0.108636E+01	0.449941E+00
81	0.595988E+00	0.164325E+00	0.109684E+01	0.272626E+00
121	0.102074E+01	0.157416E+00	0.432835E+01	0.495499E+00
169	0.262946E+00	0.378430E–01	0.163024E+01	0.154012E+00
225	0.459867E–01	0.646133E–02	0.386610E+00	0.323181E–01
289	0.161664E–02	0.377384E–03	0.164938E–01	0.179782E–02
361	0.122916E–02	0.117503E–03	0.200465E–01	0.123574E–02
441	0.181308E–02	0.232300E–03	0.287571E–01	0.183550E–02

The exact solution of this problem is given by:

$$u(x,y) = \sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4}.$$

The fourth example is the most difficult to solve using the RBF method. As the RBF approximation of function f is weak, the approximation of the solution is found with a mean-square error of 0.07%, as it is shown in Tables 10 and 11. Better results are obtained by using the polyharmonic functions method, as shown in Table 12.

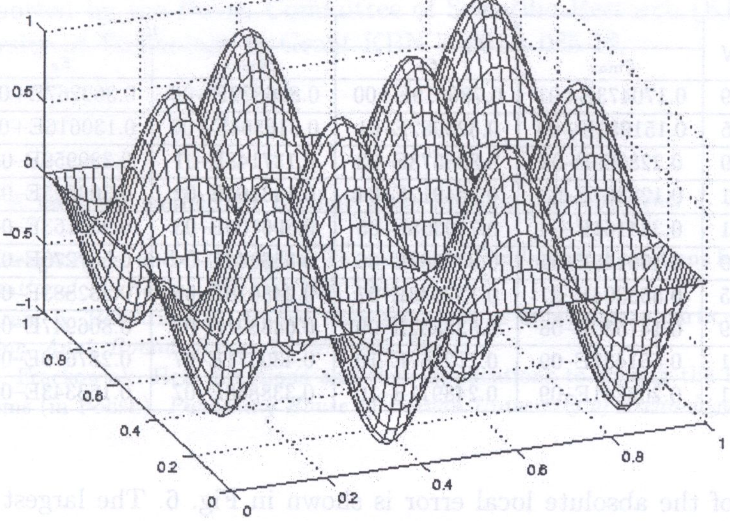

Fig. 5. The function f considered in Example 4

Table 10. Error in the approximation of function f (Example 4) using Eq. (11)

N	$\varepsilon_{\max}$	$\varepsilon_{\mathrm{ms}}$
9	.2494453	1.3925240E–02
16	.3627093	2.3694286E–02
25	.2055577	1.2584967E–02
36	.1416468	7.1109487E–03
49	9.7312033E–02	3.9964966E–03
64	6.8293795E–02	2.3214698E–03
81	4.9052790E–02	1.6569750E–03
100	3.6350735E–02	1.0619513E–03
121	2.7965140E–02	7.3561346E–04

Table 11. Error in the solution of the inhomogeneous boundary problem (Example 4) using Eq. (11)

$NC4$	$\varepsilon_{\max}$	$\varepsilon_{\mathrm{ms}}$
44	9.4121024E–03	6.7816587E–04
48	9.3666166E–03	6.9940113E–04
52	9.3953684E–03	7.1001839E–04
56	9.3951300E–03	7.2642759E–04
60	9.3854666E–03	7.3569641E–04
64	9.3674809E–03	7.2640716E–04
68	9.3659833E–03	7.2793610E–04
72	9.3794465E–03	7.2665303E–04
76	9.3651935E–03	7.1856938E–04
80	9.3597248E–03	7.4665528E–04
84	9.4242990E–03	7.3481060E–04
88	9.3083978E–03	7.3053350E–04
92	9.4419494E–03	7.2038569E–04
96	1.0760002E–02	2.8416610E–04
100	8.2345903E–03	2.0362045E–04
104	3.0615605E–02	9.1372227E–04

Table 12. Error in the approximation of function f and of the solution of the inhomogeneous boundary problem (Example 4) by the polyharmonic functions method

N	f		u	
	$\varepsilon_{\max}$	ε_s	$\varepsilon_{\max}$	ε_s
9	0.170473E+01	0.786619E+00	0.886872E+00	0.362267E+00
25	0.151951E+00	0.520982E–01	0.316564E+00	0.130616E+00
49	0.228969E–02	0.826573E–03	0.137148E–01	0.389959E–02
81	0.128367E–03	0.384912E–04	0.143264E–02	0.303525E–03
121	0.334161E–04	0.719358E–05	0.806700E–03	0.903163E–04
169	0.159148E–05	0.338289E–06	0.559346E–04	0.507279E–05
225	0.553304E–07	0.137933E–07	0.248618E–05	0.232883E–06
289	0.147530E–08	0.538832E–09	0.643891E–07	0.806947E–08
361	0.561437E–09	0.712967E–10	0.430671E–07	0.287679E–08
441	0.207241E–09	0.248911E–10	0.338868E–07	0.163343E–08

The localisation of the absolute local error is shown in Fig. 6. The largest errors occur at the corners of the region considered.

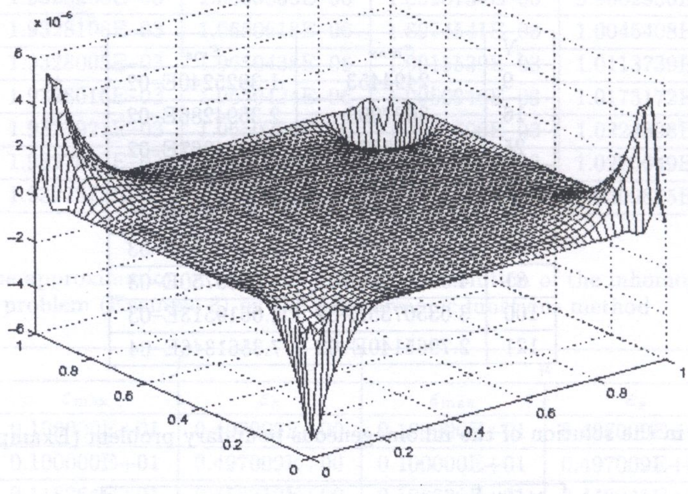

Fig. 6. The local error distribution for the solution of Example 4

4. Conclusions

The examples presented in this paper show that quite accurate approximations are obtained for the forcing function f in the Eq. (1) using radial basis functions. Such approximations yield good results for the solution of the inhomogeneous boundary problem using the method of fundamental solutions.

The second method considered is quite interesting. It represents a new approach to the solution of the problem using the inverse operator. As it has been shown, the method gives very good approximations for the solution, with one exception.

Generally speaking, both methods are adequate procedures to solve the inhomogeneous elliptic problems.

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