

The derivation of special purpose element functions using complex solution representations

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For several elasticity problems, solution representations for the displacements and stresses are available. The solution representations are given in terms of “arbitrary” complex valued functions. For any choice of the complex functions, the governing differential equations are automatically satisfied. Complex solution representations are therefore useful for applications of the Trefftz method. For the analysis of local stress concentrations, due to the local geometry of the boundary curve, such solution representations can be very helpful in the construction of appropriate series of Trefftz functions. In this paper, a few examples are given to demonstrate how to construct Trefftz functions for special purpose finite elements, which include the local solution behavior around a stress concentration or stress singularity.

Keywords: elasticity, complex solution representations, stress singularities, Trefftz functions, Trefftz-type finite elements

1. INTRODUCTION

Erich Trefftz opened new approximation possibilities with his method introduced in 1926 [1]. The Trefftz method is based on the use of a set of linearly independent trial functions which a priori satisfy the differential equation under consideration. The Trefftz method can be used for finite element and boundary element approximations. Since its introduction, a variety of papers appeared on the Trefftz method (see e.g. Almeida/Pereira [2], Dumont [3], Freitas/Ji [4], Herrera [5], Jirousek et al. [6–10], Kita/Kamiya [11], Kompis/Konkol/Vasko [12], Leitão [13], Maunder/Almeida [14, 15], Moorthy/Ghosh [16], Petrolito [17], Piltner [18–25], Qin [26], Reutsky/Tirozzi [27], Ruoff [28], Stein [29, 30], Szabo/Babuska [31], Zhang/Katsube [32], Zielinski/Zienkiewicz [33], Zielinski [34], Zienkiewicz et al. [35], Zienkiewicz/Taylor [36]). In order to utilize the idea of Trefftz for obtaining approximate solutions for differential equations, we have to construct appropriate trial functions which satisfy the governing differential equations. Depending on the geometry of local boundary curves, we need series of functions in curvilinear coordinates. Instead of working with real functions, we can have an advantage using complex functions. For example, for several problems in mechanics, complex solution representations are available and can be used with advantage. The basis of complex function methods for elasticity has been developed in the first half of the twentieth century [37–39]. After numerical methods such as the finite element method and the boundary element method became available as well as powerful computers and symbolic manipulation programs, new possibilities for utilizing complex functions for numerical approximations in mechanics became feasible. In this paper, a brief overview is given for some possibilities to construct and use complex functions in numerical approximations related to the Trefftz method and finite elements.

2. COMPLEX SOLUTION REPRESENTATIONS IN ELASTICITY

The displacements, strains and stresses for several elasticity problems can be expressed in terms of "arbitrary" functions. For example, the deflection of a Kirchhoff plate can be expressed in terms of the arbitrary functions Φ and Ψ as $w(x, y) = \text{Re}[\bar{z}\Phi(z) + \Psi(z)]$. The functions Φ and Ψ are arbitrary in the sense that with any choice for these complex functions the governing differential equation is automatically satisfied. Choosing complex functions, as for example complex power series in different systems of curvilinear coordinates enables us to construct systematically a variety of sets of approximation functions. Solution representations are available for isotropic and anisotropic thin plates under bending, for plane strain and stress problems, for the stretching and bending of thick plates [21–24], and for three-dimensional elasticity problems [25]. In this paper some strategies for plane strain and stress problems are discussed. The ideas behind the discussed strategies can be translated to several other types of problems.

3. EXAMPLE: COMPLEX SOLUTION REPRESENTATION FOR PLANE STRAIN/STRESS

The displacement components and the stresses for plane strain/stress can be written in terms of two arbitrary functions $\Phi(z)$ and $\Psi(z)$ of the complex variable $z = x + iy$ in the following form [39]:

$$2\mu u = \text{Re} \left[\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)} \right], \quad (1)$$

$$2\mu v = \text{Im} \left[\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)} \right], \quad (2)$$

$$\sigma_{xx} = \text{Re} \left[2\Phi'(z) - \bar{z}\Phi''(z) - \Psi'(z) \right], \quad (3)$$

$$\sigma_{yy} = \text{Re} \left[2\Phi'(z) + \bar{z}\Phi''(z) + \Psi'(z) \right], \quad (4)$$

$$\tau_{xy} = \text{Im} \left[\bar{z}\Phi''(z) + \Psi'(z) \right], \quad (5)$$

where

$$2\mu = E/(1 + \nu),$$

$$\kappa = \begin{cases} (3 - \nu)/(1 + \nu) & \text{for plane stress,} \\ (3 - 4\nu) & \text{for plane strain.} \end{cases} \quad (6)$$

A displacement boundary condition with given displacements \bar{u} and \bar{v} can be written as

$$\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)} = 2\mu(\bar{u} + i\bar{v}) \quad \text{on } \Gamma_u, \quad (7)$$

$$\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)} = i \int (\bar{T}_x + i\bar{T}_y) ds \quad \text{on } \Gamma_T. \quad (8)$$

When using a conformal mapping $z = f(\zeta)$ involving the complex variable $\zeta = \xi + i\eta$, one has to distinguish the derivatives

$$\Phi'(z) = \frac{d\Phi(z)}{dz} \quad (9)$$

and

$$\dot{\Phi}(\zeta) = \frac{d\Phi(\zeta)}{d\zeta}. \quad (10)$$

If Φ is assumed in the transformed domain (i.e. $\Phi = \Phi(\zeta)$), we have

$$\Phi' = \frac{\dot{\Phi}(\zeta)}{f'(\zeta)} \quad (11)$$

and

$$\Phi'' = \frac{\ddot{\Phi}(\zeta)}{f^2(\zeta)} - \dot{\Phi}(\zeta) \frac{\dot{f}(\zeta)}{f^3(\zeta)} \tag{12}$$

4. FUNCTIONS FOR FINITE ELEMENTS WITH EXTERNAL CRACKS

Some of the first numerical applications of complex functions were for linear elastic crack problems [40, 41]. Here the derivation of Trefftz functions via a complex variable representation is considered. The finite element domain in Fig. 1 can be mapped via the conformal mapping

$$\zeta = \sqrt{z} = \sqrt{r} \cos \frac{\varphi}{2} + i\sqrt{r} \sin \frac{\varphi}{2} \tag{13}$$

and the associated inverse mapping is

$$z = f(\zeta) = \zeta^2 \tag{14}$$

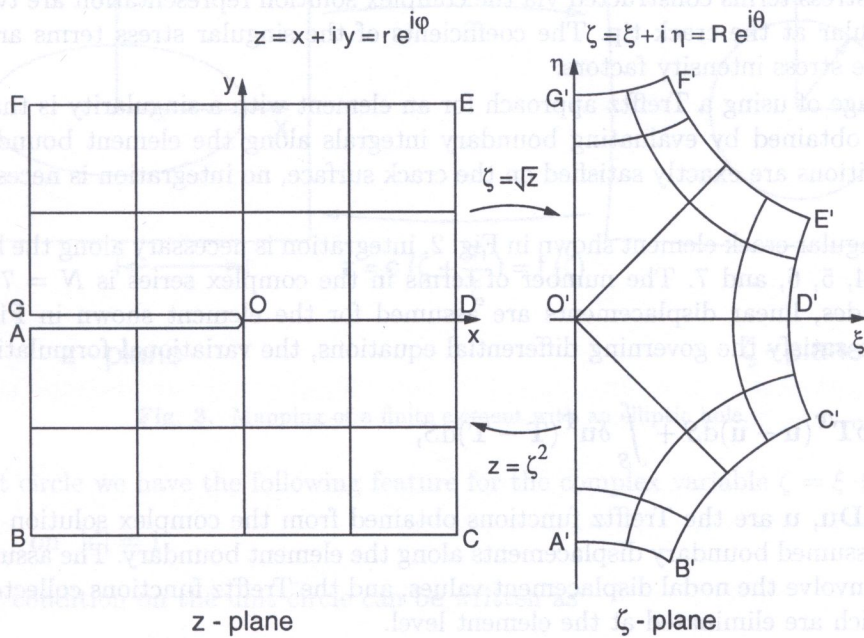


Fig. 1. Element with crack and conformal mapping

The location of the mapped crack surface is at $\xi = 0$. The displacements for the crack element are obtained from the complex solution representation of Muskhelishvili-Kolosov (1)-(5). Instead of assuming the complex functions in the original element domain, the complex functions are assumed in the transformed domain as complex power series of the complex variable $\zeta = \xi + i\eta$:

$$\Phi = \sum_j a_j \zeta^j, \tag{15}$$

$$\Psi = \sum_j b_j \zeta^j. \tag{16}$$

In order to satisfy the stress free boundary conditions on the crack surface, a relationship between the coefficients in Φ and Ψ has to be established. The stress boundary condition on the crack can

be written as

$$\overline{\Phi(\zeta)} + f(\zeta) \frac{\dot{\Phi}(\zeta)}{f(\zeta)} + \Psi(\zeta) = 0. \quad (17)$$

On the mapped crack ($\xi = 0$), we can get rid of conjugate complex terms $\overline{\zeta^j}$ by utilizing

$$\overline{\zeta^j} = (-1)^j \zeta^j \quad \text{on } \xi = 0. \quad (18)$$

For Eq. (17) we use the assumed function Φ of (15). Solving (17) for Ψ and utilizing (18), we get

$$\Psi(\zeta) = - \sum_{j=0}^N [\bar{a}_j (-1)^j + \frac{j}{2} a_j] \zeta^j. \quad (19)$$

The two functions Φ and Ψ from Eqs. (15) and (19) substituted into the Muskhelishvili-Kolosov representation guarantees i) the satisfaction of the equilibrium equations, and ii) the satisfaction of the stress-free boundary conditions on the crack.

Among the stress terms constructed via the complex solution representation are two stress terms which are singular at the crack tip. The coefficients of the singular stress terms are used for the definition of the stress intensity factors.

The advantage of using a Trefftz approach for an element with a singularity is that the stiffness matrix can be obtained by evaluating boundary integrals along the element boundary. Since the boundary conditions are exactly satisfied on the crack surface, no integration is necessary along the crack.

For the triangular crack element shown in Fig. 2, integration is necessary along the line connecting nodes 1, 2, 3, 4, 5, 6, and 7. The number of terms in the complex series is $N = 7$. Between two neighboring nodes, linear displacements are assumed for the element shown in Fig. 2. Utilizing functions which satisfy the governing differential equations, the variational formulation reduces to

$$\delta \Pi = - \int_S \delta \mathbf{T}^T (\mathbf{u} - \tilde{\mathbf{u}}) dS + \int_S \delta \mathbf{u}^T (\mathbf{T} - \bar{\mathbf{T}}) dS, \quad (20)$$

where $\mathbf{T} = \mathbf{nEDu}$, \mathbf{u} are the Trefftz functions obtained from the complex solution representation and $\tilde{\mathbf{u}}$ are the assumed boundary displacements along the element boundary. The assumed boundary displacements involve the nodal displacement values, and the Trefftz functions collected in \mathbf{u} involve parameters which are eliminated at the element level.

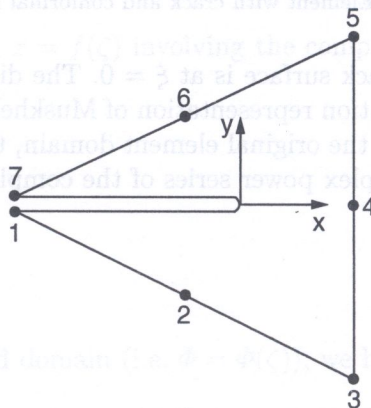


Fig. 2. Triangular crack element

5. FUNCTIONS FOR FINITE ELEMENTS WITH CIRCULAR OR ELLIPTICAL HOLES

For the construction of Trefftz functions for a finite element with an elliptical hole, the finite element domain is mapped to a new domain (Fig. 3) in which the boundary of the elliptical hole is the boundary of a unit circle (radius $R = 1$). The conformal mapping

$$z = f(\zeta) = c \left(\zeta + \frac{m}{\zeta} \right), \tag{21}$$

involves the semiaxes a and b of the ellipse:

$$c = \frac{a + b}{2}, \tag{22}$$

$$m = \frac{a - b}{a + b}. \tag{23}$$

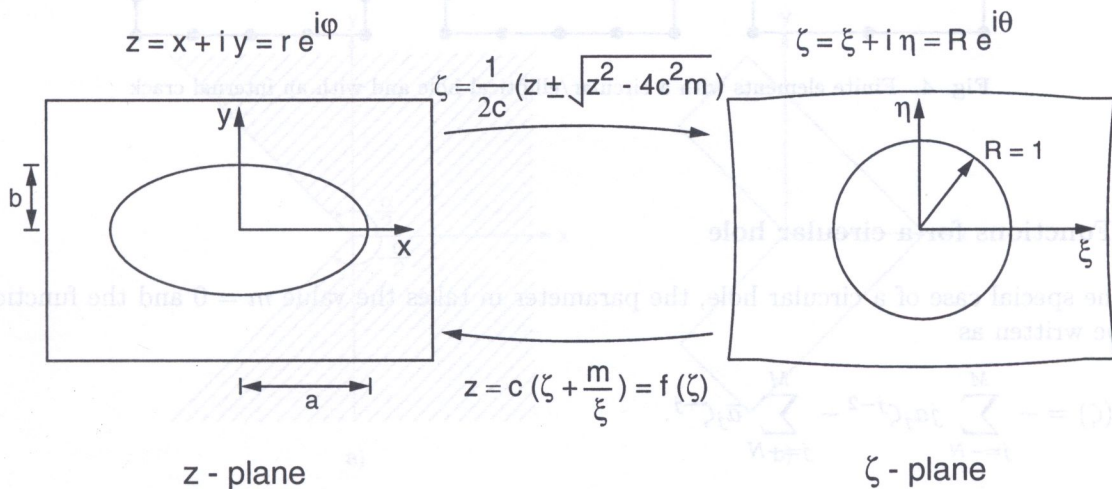


Fig. 3. Mapping of a finite element with an elliptic hole

On the unit circle we have the following feature for the complex variable $\zeta = \xi + i\eta$:

$$\overline{\zeta^j} = \zeta^{-j} \quad \text{on } |\zeta| = 1. \tag{24}$$

The boundary condition on the unit circle can be written as

$$\Psi(\zeta) = -\overline{\Phi(\zeta)} - \overline{f(\zeta)} \frac{\dot{\Phi}(\zeta)}{f(\zeta)} \quad \text{on } |\zeta| = 1. \tag{25}$$

Because the derivative of f of the mapping function is not a constant for a general ellipse, the following substitution turns out to be convenient:

$$\Psi(\zeta) = \frac{\dot{\chi}(\zeta)}{f(\zeta)}. \tag{26}$$

The boundary condition becomes

$$\dot{\chi}(\zeta) = -\dot{f}(\zeta) \overline{\Phi(\zeta)} - \overline{f(\zeta)} \dot{\Phi}(\zeta) \quad \text{on } |\zeta| = 1. \tag{27}$$

With a chosen Φ of the form

$$\Phi(\zeta) = \sum_{j=-N}^M a_j \zeta^j \tag{28}$$

and utilizing the feature (24) on the unit circle, the expression for $\dot{\chi}$ is obtained as

$$\dot{\chi}(\zeta) = -c \sum_{j=-N}^M \bar{a}_j \zeta^{-j} + cm \sum_{j=-N}^M \bar{a}_j \zeta^{-j-2} - c \sum_{j=-N}^M ja_j \zeta^{j-2} - cm \sum_{j=-N}^M ja_j \zeta^j. \tag{29}$$

For a general ellipse the function Ψ is obtained from the substitution (26).

For the first example element shown in Fig. 4, the summation limits are chosen as $N = 4$ and $M = 4$, and between two adjacent nodes linear boundary displacements are assumed. For the second element and the third element in Fig. 4, the limits are chosen as $N = 8$ and $M = 8$, and boundary displacements are chosen as piecewise quadratic functions.

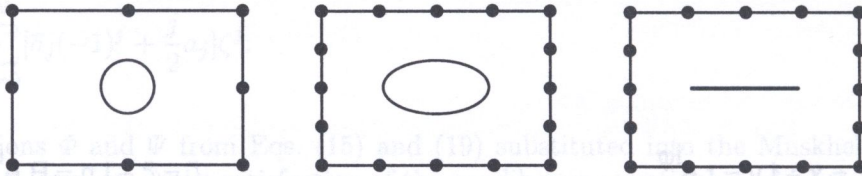


Fig. 4. Finite elements with a circular/elliptical hole and with an internal crack

5.1. Functions for a circular hole

For the special case of a circular hole, the parameter m takes the value $m = 0$ and the function Ψ can be written as

$$\Psi(\zeta) = - \sum_{j=-N}^M ja_j \zeta^{j-2} - \sum_{j=-N}^M \bar{a}_j \zeta^{-j}. \tag{30}$$

The pair of functions Φ and Ψ from Eqs. (28) and (30) guarantee the satisfaction of both the equilibrium equations and the boundary conditions on the hole boundary.

5.2. Functions for the case of constant pressure

For the case of constant pressure p on the hole boundary, the following pair of complex functions can be derived:

$$\Phi(\zeta) = 0, \tag{31}$$

$$\Psi(\zeta) = -pc \left(\frac{1}{\zeta} + m\zeta \right). \tag{32}$$

6. STRESSES IN THE NEIGHBORHOOD OF RE-ENTRANT CORNERS

For the case of plane strain and plane stress problems the local behavior near a corner is illustrated. Using only real functions, Williams analyzed the local solution behavior in corner regions [42]. Here we want to construct the displacements and stresses in the neighborhood of a corner with the complex function representation of Muskhelishvili–Kolosov (1)–(5). There is an elegant way to express the two complex functions for this problem.

The advantage of using the complex representation (1)–(5) is that with any choice of complex functions $\Phi(z)$ and $\Psi(z)$ the equilibrium equations are automatically satisfied. In order to describe

the local solution behavior in the neighborhood of a corner with the angle α defined in Fig. 5 we need functions which satisfy the stress free boundary conditions

$$\sigma_{\varphi\varphi} = 0 \quad \text{on} \quad \varphi = \pm \frac{\alpha}{2}, \tag{33}$$

$$\tau_{r\varphi} = 0 \quad \text{on} \quad \varphi = \pm \frac{\alpha}{2}. \tag{34}$$

For an arbitrary angle α it is not possible to use power series of the form

$$\Phi(z) = \sum_j a_j z^j, \tag{35}$$

$$\Psi(z) = \sum_j b_j z^j \tag{36}$$

and to be able to satisfy the boundary conditions. However, if instead of integer exponents we admit real or even complex exponents it becomes possible to satisfy the boundary conditions (33)–(34).

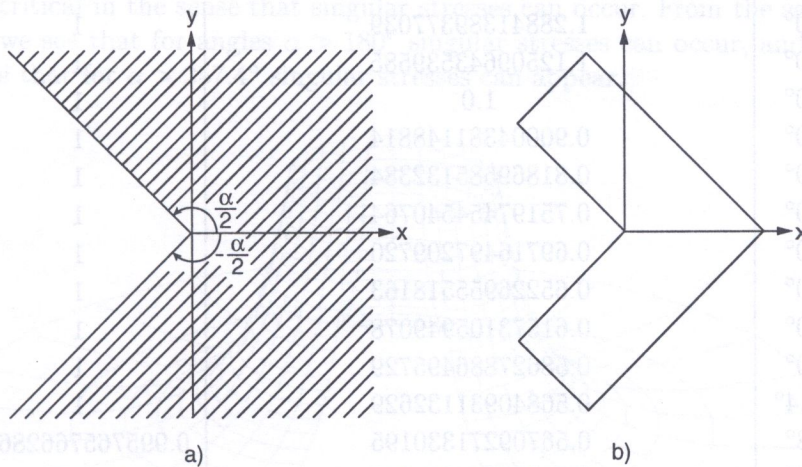


Fig. 5. a) Domain with a re-entrant corner; b) Finite element with re-entrant corner

For a given angle α there is a series of possible exponents for the power series of the complex functions Φ and Ψ . Assuming the first complex function in the form

$$\begin{aligned} \Phi(z) &= \sum_{j=0}^{\infty} a_j z^{\lambda_j} + \sum_{j=0}^{\infty} \bar{a}_j z^{\bar{\lambda}_j} + \sum_{j=0}^{\infty} b_j z^{\omega_j} + \sum_{j=0}^{\infty} \bar{b}_j z^{\bar{\omega}_j} \\ &= \sum_{j=0}^{\infty} [\alpha_j (z^{\lambda_j} + z^{\bar{\lambda}_j}) + \beta_j (z^{\lambda_j} - z^{\bar{\lambda}_j})] + \sum_{j=0}^{\infty} [\gamma_j (z^{\omega_j} - z^{\bar{\omega}_j}) + \delta_j (z^{\omega_j} + z^{\bar{\omega}_j})], \end{aligned} \tag{37}$$

where

$$a_j = \alpha_j + i\beta_j, \tag{38}$$

$$b_j = \gamma_j + i\delta_j, \tag{39}$$

one can find the second function Ψ and the unknown exponents from the boundary conditions (33)–(34). The function Ψ can be found in the form

$$\begin{aligned} \Psi(z) &= - \sum_{j=0}^{\infty} a_j [e^{i\lambda_j\alpha} + \lambda_j e^{i\alpha}] z^{\lambda_j} - \sum_{j=0}^{\infty} \bar{a}_j [e^{i\bar{\lambda}_j\alpha} + \bar{\lambda}_j e^{i\alpha}] z^{\bar{\lambda}_j} \\ &+ \sum_{j=0}^{\infty} b_j [e^{i\omega_j\alpha} - \omega_j e^{i\alpha}] z^{\omega_j} + \sum_{j=0}^{\infty} \bar{b}_j [e^{i\bar{\omega}_j\alpha} - \bar{\omega}_j e^{i\alpha}] z^{\bar{\omega}_j}. \end{aligned} \tag{40}$$

The exponents λ_j and ω_j are solutions of the characteristic equations

$$\sin \alpha \lambda_j + \lambda_j \sin \alpha = 0, \quad (41)$$

$$\sin \alpha \omega_j - \omega_j \sin \alpha = 0. \quad (42)$$

Table 1. Exponents λ_1 and ω_1 for different values of α

α	λ_1	ω_1
20°	12.07947991720516 + i 6.38438830561358	1
30°	8.06296525882784 + i 4.20286708520903	1
60°	4.05932901215137 + i 1.95204994722751	1
90°	2.73959335632460 + i 1.11902453434242	1
120°	2.09413910919242 + i 0.60458500270356	1
150°	1.53386000277759	1
160°	1.28841389377039	1
170°	1.12509643539585	1
180°	1.0	1
190°	0.90004381148814	1
200°	0.81869585132384	1
210°	0.75197454540764	1
220°	0.69716497209720	1
230°	0.65226955518163	1
240°	0.61573105949078	1
250°	0.58627886495729	1
257.4°	0.56840931132629	1
258°	0.56709271330195	0.99576576628600
260°	0.56283948048168	0.98047492545310
270°	0.54448373678246	0.90852918984610
280°	0.53039571912977	0.84343956892930
290°	0.51985430311392	0.78444055297409
300°	0.51222136116051	0.73090074151295
310°	0.50693284228647	0.68229483030706
320°	0.50349048318478	0.63818247129336
330°	0.50145300871355	0.59819184961408
340°	0.50042637542606	0.56200654961948
350°	0.50005298712644	0.52935473834138
360°	0.5	0.5

In Table 1, solutions for the characteristic equations are given for different angles α . Figures 7 and 8 show the smallest exponents as a function of the angle α . For the angle $\alpha = 270^\circ$ (Fig. 5), for example, the first non-zero exponents (Table 1) are

$$\lambda_1 = 0.54448373678246 \quad (43)$$

and

$$\omega_1 = 0.90852918984610. \quad (44)$$

The terms involving λ_1 and ω_1 lead to singular stresses.

For other angles α , singular stresses are possible if the exponents satisfy the following relationships:

$$\text{Re}[\lambda_j] < 1, \tag{45}$$

$$\text{Re}[\omega_j] < 1. \tag{46}$$

Using polar coordinates r and φ the sum of the normal stresses which is an invariant can be written for the first terms as

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\varphi\varphi} &= 4\text{Re}[\alpha_1 \lambda_1 z^{\lambda_1-1} + i\delta_1 \omega_1 z^{\omega_1-1}] \\ &= 8[\alpha_1 \lambda_1 r^{\lambda_1-1} \cos(\lambda_1 - 1)\varphi - \delta_1 \omega_1 r^{\omega_1-1} \sin(\omega_1 - 1)\varphi]. \end{aligned} \tag{47}$$

It is seen that for exponents λ_1 and ω_1 satisfying Eqs. (41) and (42) the sum of the normal stresses in Eq. (47) becomes infinite for $r = 0$. In Fig. 6 a plot of the term involving $\lambda_1 = 0.54448373678246$ is shown. From the plots of the smallest exponents which are solutions of (41) or (42) we find which corner angles are critical in the sense that singular stresses can occur. From the sequence of values for λ_j in Table 1 we see that for angles $\alpha > 180^\circ$ singular stresses can occur, and the sequence of ω_j values indicates that for $\alpha > 257.4^\circ$ singular stresses can appear.

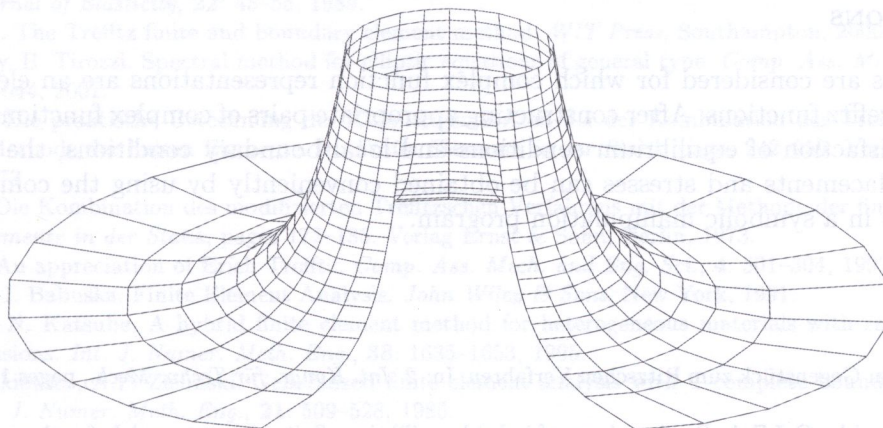


Fig. 6. Singular term for $\sigma_{xx} + \sigma_{yy}$ in the neighborhood of a corner

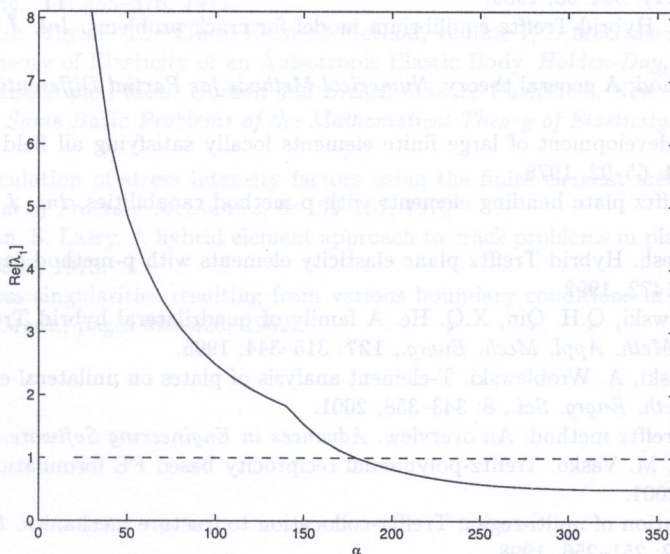


Fig. 7. Exponent λ_1 as a function of the angle α

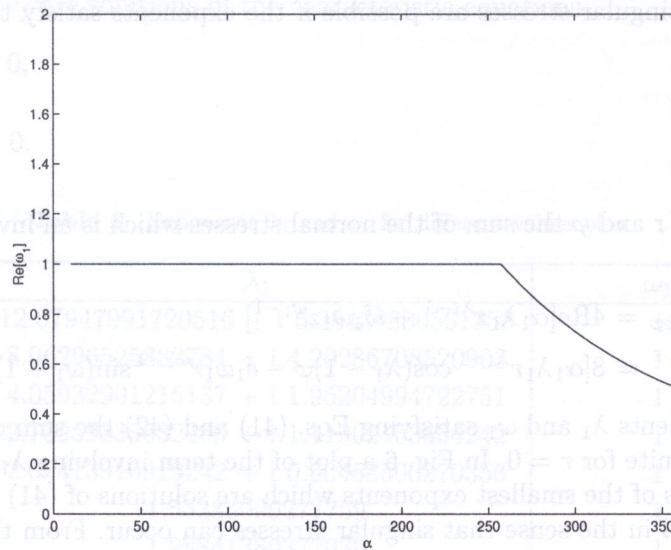


Fig. 8. Exponent ω_1 as a function of the angle α

7. CONCLUSIONS

A few examples are considered for which complex function representations are an elegant tool for constructing Trefftz functions. After constructing appropriate pairs of complex functions which guarantee both satisfaction of equilibrium conditions and local boundary conditions, the real physical quantities displacements and stresses can be obtained conveniently by using the complex solution representations in a symbolic manipulation program.

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