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Variationally-based hybrid boundary element methods

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The hybrid stress boundary element method (HSBEM) was introduced in 1987 on the basis of the Hellinger-Reissner potential, as a generalization of Pian's hybrid finite element method. This new two-field formulation makes use of fundamental solutions to interpolate the stress field in the domain of an elastic body, which ends up discretized as a superelement with arbitrary shape and arbitrary number of degrees of freedom located along the boundary. More recently, a variational counterpart – the hybrid displacement boundary element method (HDBEM) – was proposed, on the basis of three field functions, with equivalent advantages. The present paper discusses these methods as well as the traditional, collocation boundary element method (CBEM). The mechanical properties of the resulting matrix equations are investigated and a series of concepts in both HDBEM and CBEM that have not been properly considered by previous authors, particularly in which concerns body forces, are redefined. This is not a review paper, but rather a theoretical, comparative analysis of three methods, with many physical considerations, some innovations and a few academic illustrations.

Keywords: Boundary element methods, generalized inverse matrices, variational methods.

1. Introduction

The resultant equations of the conventional boundary element method (CBEM) cannot be derived through variational considerations. Any energetically consistent formulation of problems for which the principle of superposition (and therefore Betti's reciprocal theorem) is valid must yield symmetric matrices for any finite discretization, whether integral equations are used or not. The author introduced in 1987 a proper and to a certain extent new formulation of the boundary element method to demonstrate this assertion [1, 2]. It was based on the generalized expression of the total potential energy and clarified the discussion of the symmetry characteristics of the resultant equations: symmetry or nonsymmetry is a matter of adequate or inadequate variational treatment of the boundary conditions. Moreover, it was shown that the variationally consistent generalized displacement formulation introduced is equivalent to a formulation based on the Hellinger–Reissner potential, exactly as Pian [3] had developed for finite elements. In allusion to Pian's work this new method was baptized *the hybrid boundary element method* (see Oden and Reddy [4] for the exact meaning of the word *hybrid*). Since the method is based on a stress field assumption, a more specific title might be the hybrid *stress* boundary element method – HSBEM.

A few years after the introduction of the HSBEM, De Figueiredo and Brebbia [5] proposed a variational counterpart, properly called the hybrid *displacement* boundary element method – HDBEM. The HDBEM is equally consistent and presents the same computational characteristics of the HSBEM, although based on a different (three-field) variational principle. Making use of his experience in dealing with the HSBEM, the present author endeavored to make a conceptual assessment of the HDBEM [6]. That work focused in particular on the identification and exploitation of the inherent spectral properties of matrices involved in the formulation and solution of general problems. In this way the initial work of De Figueiredo and Brebbia was completed.

This paper starts with a reformulation of the conventional boundary element method (CBEM) prompted by the fact that arbitrary rigid body displacements, inherent to any fundamental solution, should have no influence on the accuracy of the final results [7]. Although this revisiting of the CBEM is the product of a recent investigation, it seems more didactical to lay out the paper in the reverse chronological order of the author's developments, going from the CBEM to the HDBEM and finally to the HSBEM, the actual starting point.

2. Some basic considerations on the fundamental solutions

Consider the fundamental solution of a generic three-dimensional elasticity problem, expressed in terms of displacements u_i^* measured at a given point for a given coordinate direction i of the domain, caused by some arbitrary, concentrated force p_m^* acting according to a given degree of freedom m (the index m characterizes both a point and a direction in the domain):

$$u_i^* = u_{im}^* p_m^* + u_{is}^r r_s \equiv (u_{im}^* + u_{is}^r C_{sm}) p_m^*. \tag{1}$$

This fundamental solution, as characterized by the superscript "*", is usually given in the literature by the function u_{im}^* alone, implicitly related to unitary forces p_m^*. The complete representation of Eq. (1) is both mathematically and physically more adequate, since it is stated for an arbitrary (not unitary) concentrated force p_m^* and a term is added to take into account the arbitrary rigid body displacements, as denoted by the superscript r. In the rigid body displacement functions u_{is}^r, s refers to the rigid body displacement being interpolated. The quantities r_s are arbitrary constants, which may be correlated to the concentrated forces p_m^* through some arbitrary matrix C_{sm} of constants. In this paper, subscripts m and n refer to degrees of freedom of discretized quantities; subscripts s and t refer to rigid body displacements; and subscripts i and j are related to the coordinate directions.

The stresses at a given point of the domain are obtained from Eq. (1) as,

$$\sigma_{ij}^* = \sigma_{ijm}^* p_m^* \text{ such that } \sigma_{ji,j}^* = \sigma_{ijm,j}^* p_m^* = 0 \text{ in } \Omega \tag{2}$$

as a property of a fundamental solution, for a domain Ω that does not include the points of application of p_m^*. If some domain Ω_0 should comprise the point of application of a concentrated force p_m^*, then:

$$\int_{\Omega_0} \sigma_{ijm,j}^* \mathrm{d}\Omega = -\delta_{im} = \begin{cases} -1 & \text{if } i \text{ and } m \text{ refer to the same degree of freedom,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

From the stresses in Eq. (2) one derives the traction forces along the boundary Γ as

$$\sigma_{ijm}^* \eta_j p_m^* \equiv p_{im}^* p_m^*, \tag{4}$$

where η_j are the director cosines of the outward normal to the boundary.

The aim of this short outline was to introduce the terminology needed in the rest of the paper. As it is presented above, one is dealing with Green's functions, the singularity of which is required to formulate an integral statement, the Somigliana identity, as the basis of the CBEM. For the development of variational methods, on the other hand, a fundamental solution may be based on non-singular (polynomial) functions, as in Pian's hybrid finite element method or in the Trefftz methods, in general. However, the use of singular functions simplifies the whole formulation and ensures that the resultant matrix equations are well conditioned – at the cost of dealing with singular and improper integrals. The combination of singular, non-singular functions and some special functions, e.g. the Westergaard stress function in fracture mechanics, may be of advantage when dealing with some particular stress gradients [8].

3. The traditional boundary element equation

The results obtained in a two-dimensional (traditional) boundary element formulation vary with the scale chosen to describe a problem when approximations are involved. The researchers relate this fact to the presence of the logarithm term in the fundamental solution. This is only the more conspicuous aspect of the fact, which is also verified in a three-dimensional formulation, that adding a constant to a fundamental solution does affect the final results and could even contribute to ill-conditioning. It is also well known that, differently from the finite element method and independently of computational precision, the (traditional) boundary element formulation yields non-equilibrated solutions for both two-dimensional and three-dimensional problems, unless the results coincide with the analytical ones [9]. A number of research works on these subjects have been published in the last years, but they do not report convincing results. The most complete investigation to date is possibly the one by Telles and De Paula [10].

The collocation boundary element method is revisited in this Section. It is shown that a matrix singularity (or, expressing it in a more suitable way, a well established and understood matrix spectral property, that may arise in a formulation independently of the fact that the underlying fundamental solution and the consequent boundary integral equation involve some singularity) is a welcome property to be taken advantage of, as it notoriously occurs in case of the matrix $\mathbf{H}$ of the conventional boundary element method. Matrix $\mathbf{G}$ of a consistently formulated boundary element method is or should also be singular whenever it is obtained properly. This is a conceptually welcome feature, as it will be demonstrated presently. The present outline is based on [7], with some slight changes in terminology and improvements concerning the computationally adequate consideration of body forces.

For the derivation of the collocation boundary element method, one may start with the weighted residual method, among other possibilities, and write down the statement

$$u_m = \int_\Gamma \left(u^*_{im} t_i - p^*_{im} u_i\right) \mathrm{d}\Gamma + \int_\Omega u^*_{im} b_i \mathrm{d}\Omega + \left(\int_\Gamma u^r_{is} t_i \mathrm{d}\Gamma + \int_\Omega u^r_{is} b_i \mathrm{d}\Omega\right) C_{sm} \tag{5}$$

for a weighting function given by the fundamental solution expressed by Eqs. (1) and (4). It relates the displacement u_m of a point in the interior of the elastic body with known boundary displacements u_i, boundary traction forces t_i and body forces b_i. Considering that the traction forces are in equilibrium with the body forces, the term in brackets multiplying the constants C_{sm} is identically null,

$$\int_\Gamma u^r_{is} t_i \mathrm{d}\Gamma + \int_\Omega u^r_{is} b_i \mathrm{d}\Omega \equiv 0 \tag{6}$$

and Eq. (5) ends up as Somigliana's identity,

$$u_m = \int_\Gamma \left(u^*_{im} t_i - p^*_{im} u_i\right) \mathrm{d}\Gamma + \int_\Omega u^*_{im} b_i \mathrm{d}\Omega \tag{7}$$

which is the basis of the direct, collocation boundary element method.

To derive the traditional boundary element method, one assumes that both displacements u_i and traction forces t_i are approximated along the boundary in terms of interpolation functions multiplying some nodal parameters d_n and traction force intensities t_l, respectively:

$$\left.\begin{array}{rcl} u_i & = & u_{in} d_n \\ t_i & = & t_{il} t_l \end{array}\right\} \quad \text{along } \Gamma. \tag{8}$$

Usually, the same interpolation functions are assumed for displacements and traction forces, $u_{in} \equiv t_{in}$, although there is no mechanical basis for this simplification. It is worth observing that one may have more traction force intensity parameters t_l than nodal displacements d_n [11].

Now, instead of using Somigliana's identity (7), the more general Eq. (5), together with Eqs. (8), are used to express a set of displacement compatibility equations at points distributed all along the boundary,

$$\left[\int_\Gamma p^*_{im} u_{in} \mathrm{d}\Gamma + \delta_{mn}\right] d_n \tag{9}$$
$$= \left[\int_\Gamma u^*_{im} t_{il} \mathrm{d}\Gamma\right] t_l + \left\{\int_\Omega u^*_{im} b_i \mathrm{d}\Omega\right\} + \left(\left[\int_\Gamma u^r_{is} t_{il} \mathrm{d}\Gamma\right] t_l + \int_\Omega u^r_{is} b_i \mathrm{d}\Omega\right) C_{sm}$$

in terms of some nodal parameters d_n and some traction force intensities t_l, which are in part known and in part unknown, leading to a system with as many equations as unknowns. In Eq. (9), the Kronecker delta δ_{mn} results from the identity $u_m \equiv \delta_{mn} d_n$, for displacements u_m evaluated at nodal points.

In matrix notation, Eq. (9) is written as,

$$\mathbf{Hd} = \mathbf{Gt} + \mathbf{b}^* + \mathbf{C}^{\mathrm{T}}(\mathbf{R}^{\mathrm{T}}\mathbf{t} + \mathbf{b}^r), \tag{10}$$

where $\mathbf{C}^{\mathrm{T}}$ replaces $\mathbf{C}$ in the notation used in [7]. The terms $\mathbf{d} \equiv d_n$ and $\mathbf{t} \equiv t_l$ in Eq. (10) are vectors corresponding to boundary displacement and traction parameters, respectively. The kinematic transformation matrix,

$$\mathbf{H} \equiv H_{mn} = \int_\Gamma p^*_{im} u_{in} \mathrm{d}\Gamma + \delta_{mn} \tag{11}$$

is defined by the first term in Eq. (9), assuming that the singularities of the boundary integral have been properly dealt with, and observing also Eqs. (2) and (3). The flexibility-like matrix,

$$\mathbf{G} \equiv G_{ml} = \int_\Gamma u^*_{im} t_{il} \mathrm{d}\Gamma \tag{12}$$

is defined by the second term of Eq. (9), an improper integral that may also present some quasi-singularities [12]. The role of matrix,

$$\mathbf{R} \equiv R_{ls} = \int_\Gamma u^r_{is} t_{il} \mathrm{d}\Gamma \tag{13}$$

is discussed in Section 3.1.

In Eq. (10) there are also two vectors of equivalent nodal displacements due to body forces,

$$\mathbf{b}^* \equiv b^*_m = \int_\Omega u^*_{im} b_i \mathrm{d}\Omega, \tag{14}$$

$$\mathbf{b}^r \equiv b^r_s = \int_\Omega u^r_{is} b_i \mathrm{d}\Omega \tag{15}$$

that require a lengthy discussion on their evaluation in terms of boundary integrals, which cannot be undertaken in the context of the present paper [13, 14]. However, assuming that one can solve the equilibrium differential equation,

$$\sigma^b_{ji,j} + b_i = 0 \text{ in } \Omega, \tag{16}$$

for σ^b_{ji} as an arbitrary (as simple and convenient as possible) particular solution which exists analytically, with corresponding particular displacements u^b_i, it is possible to rewrite Eqs. (14) and (15) as [7],

$$\mathbf{b}^* \equiv b^*_m = -\int_\Gamma \sigma^b_{ji} \eta_j u^*_{im} \mathrm{d}\Gamma + \int_\Gamma p^*_{im} u^b_i \mathrm{d}\Gamma + \delta_{im} u^b_i, \tag{17}$$

$$\mathbf{b}^r \equiv b^r_s = -\int_\Gamma \sigma^b_{ji} \eta_j u^r_{is} \mathrm{d}\Gamma \tag{18}$$

in terms of boundary integrals. Further, considering that the particular solution may be approximated on the boundary in terms of the assumed interpolation functions introduced in Eq. (8), with displacement parameters $\mathbf{d}^b \equiv d_n^b$ equal to u_i^b evaluated at nodal points n, and traction parameters $t_l^b \equiv \sigma_{ji}^b \eta_j$ evaluated at element extremities l, such that,

$$\left.\begin{array}{lcl} u_i^b & \approx & u_{in} d_n^b \\ \sigma_{ji}^b \eta_j & \approx & t_{il} t_l^b \end{array}\right\} \quad \text{along } \Gamma \tag{19}$$

one may rewrite Eqs. (17) and (18) as,

$$\mathbf{b}^* = -\mathbf{G}\mathbf{t}^b + \mathbf{H}\mathbf{d}^b, \tag{20}$$

$$\mathbf{b}^r = -\mathbf{R}^{\mathrm{T}}\mathbf{t}^b \tag{21}$$

using the definitions of matrices $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{R}$ in Eqs. (11), (12) and (13), respectively. As a consequence, Eq. (10) simplifies to

$$\mathbf{H}\left(\mathbf{d} - \mathbf{d}^b\right) = \left(\mathbf{G} + \mathbf{C}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\right)\left(\mathbf{t} - \mathbf{t}^b\right). \tag{22}$$

This is, to the author's best knowledge, a novel improved expression for the equations of the collocation boundary element method.

3.1. Constructing a spectrally admissible matrix G

Unlike Somigliana's identity in Eq. (7), Eq. (9) was obtained for approximated values of the traction forces t_i, as given by the second of Eqs. (8). If one wants to enforce equilibrium, then the complementary condition,

$$\int_\Gamma u_{is}^r t_{il} \mathrm{d}\Gamma t_l + \int_\Omega u_{is}^r b_i \mathrm{d}\Omega \equiv 0 \tag{23}$$

must hold in Eq. (9). In matrix notation, as given either in Eq. (10) or (22), one obtains:

$$\mathbf{R}^{\mathrm{T}}\mathbf{t} + \mathbf{b}^r = \mathbf{0} \quad \text{or} \quad \mathbf{R}^{\mathrm{T}}\left(\mathbf{t} - \mathbf{t}^b\right) = \mathbf{0}. \tag{24}$$

Consider a rectangular matrix $\mathbf{Z}$, the columns of which are an orthogonal basis of the columns of $\mathbf{R}$, i. e. $\mathbf{Z}^{\mathrm{T}}\mathbf{Z} = \mathbf{I}$ and $\left(\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\left(\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right) = \mathbf{Z}\mathbf{Z}^{\mathrm{T}}$. The idempotent matrix $\mathbf{Z}\mathbf{Z}^{\mathrm{T}}$ is the *orthogonal projector* on the space of the inadmissible, unbalanced traction force parameters $\mathbf{t}$ [15]. For elasticity problems, the rigid body displacement functions u_{is}^r may be defined in an infinite number of ways. However, the resulting idempotent matrix $\mathbf{Z}\mathbf{Z}^{\mathrm{T}}$ is unique. Then, it follows from the definition of $\mathbf{Z}$ that,

$$\mathbf{R} = \mathbf{Z}\boldsymbol{\lambda} \tag{25}$$

in which $\boldsymbol{\lambda}$ is a non-singular square matrix readily obtained as:

$$\boldsymbol{\lambda} = \mathbf{Z}^{\mathrm{T}}\mathbf{R}. \tag{26}$$

If the traction force parameters $\mathbf{t}$ satisfy Eq. (24), a condition for Eq. (10) to be valid, it follows from Eqs. (25) and (26) that:

$$\mathbf{Z}^{\mathrm{T}}\mathbf{t} + \boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^r = \mathbf{0} \quad \text{or} \quad \mathbf{Z}^{\mathrm{T}}\left(\mathbf{t} - \mathbf{t}^b\right) = \mathbf{0}. \tag{27}$$

Pre-multiplying either of the equations above by $\mathbf{Z}$ and subtracting $\mathbf{t}$ from both sides yields the condition that $\mathbf{t}$ must satisfy to ensure the validity of Eqs. (10) and (22):

$$\mathbf{t} = \left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\mathbf{t} - \mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^r \quad \text{or} \quad \left(\mathbf{t} - \mathbf{t}^b\right) = \left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\left(\mathbf{t} - \mathbf{t}^b\right). \tag{28}$$

If this relationship is valid, then Eq. (10) should be rewritten as

$$\mathbf{H}\mathbf{d}=\mathbf{G}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\mathbf{t}+\left(\mathbf{b}^{*}-\mathbf{G}\mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^{r}\right) \tag{29}$$

or:

$$\mathbf{H}\mathbf{d}=\mathbf{G}_{a}\mathbf{t}+\mathbf{b}_{a}. \tag{30}$$

Similarly, Eq. (22) is rewritten as,

$$\mathbf{H}\left(\mathbf{d}-\mathbf{d}^{b}\right)=\mathbf{G}_{a}\left(\mathbf{t}-\mathbf{t}^{b}\right) \tag{31}$$

in which

$$\mathbf{G}_{a}\equiv\mathbf{G}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right) \tag{32}$$

is the *admissible* part of the matrix $\mathbf{G}$, obtained using the orthogonal projector given by $\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)$, and $\mathbf{b}_{a}\equiv\left(\mathbf{b}^{*}-\mathbf{G}\mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^{r}\right)$, in Eq. (30), is a vector of *admissible* nodal displacements related to the body forces. Eq. (31) seems more elegant than Eq. (30), as the vectors corresponding to body forces affect equally both sides of the equation.

An alternative way of arriving at Eqs. (30) or (31) is to attempt to obtain matrix $\mathbf{C}\equiv C_{sm}$, in Eqs. (9) and (10), in such a way that, in the absence of body forces, the nodal displacements equivalent to any set of inadmissible traction force parameters, spanned by the basis $\mathbf{Z}$, are equal to zero:

$$\left(\mathbf{G}+\mathbf{C}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\right)\mathbf{Z}=\mathbf{0}. \tag{33}$$

Then, making use of Eq. (25), one expresses the constants $\mathbf{C}$ as:

$$\mathbf{C}^{\mathrm{T}}=-\mathbf{G}\mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}. \tag{34}$$

Substitution of $\mathbf{C}$ into Eqs. (10) or (22), according to its expression above, yields the same Eqs. (30) or (31), respectively.

The admissible matrix $\mathbf{G}_a$, as defined in Eq. (32), is singular. It is worth establishing that,

$$\operatorname{rank}\left(\mathbf{G}_{a}\right)=\operatorname{rank}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right) \tag{35}$$

a feature that can be inferred physically. In fact, the matrix $\mathbf{G}$ is a flexibility-type transformation matrix, which must always yield some non-trivial nodal displacement vector from any set of traction force parameters $\mathbf{t}$, if one is dealing with an elastic body. Then, owing to this physical property, rank($\mathbf{G}$) should be equal to the number of degrees of freedom of the discretized model. However, depending on the set of rigid body displacement functions u_{is}^{r} that appears in the definition of the fundamental solution, as given in Eq. (1), some ill conditioning may occur. Regardless of the condition of matrix $\mathbf{G}$, the rank of $\mathbf{G}_a$ is always well defined according to Eq. (35), since $\mathbf{G}_a$ is, by construction, independent of the rigid body displacement functions u_{is}^{r}. The conventional collocation boundary element formulation relies on the hope that matrix $\mathbf{G}$ does not lead to ill conditioning.

All considerations in the present paper are based on the effectively reliable premiss expressed by Eq. (35). Observe that the matrix $\mathbf{G}$ is not necessarily a square matrix, as the traction force intensity parameters t_l, as surface attributes, may outnumber the number of degrees of freedom of the problem. Nonetheless, if either Eq. (30) or (31) is to be solved for some well-posed problem, one always may rearrange the rows of matrices $\mathbf{H}$ and $\mathbf{G}$ in order to arrive at a system of equations of the form,

$$\mathbf{A}\mathbf{x}=\mathbf{y}, \tag{36}$$

where vector $\mathbf{y}$ collects the known displacement and traction force parameters, besides the body force vector, and $\mathbf{A}$ is a square non-singular matrix for the solution of the unknown terms of d_n and t_l contained in the vector $\mathbf{x}$.

Observe that the governing matrix equation of the conventional boundary element method, be it with the matrix $\mathbf{G}$ or in the improved spectral formulation given by Eqs. (30) or (31), is not completely consistent, owing to the redundant, mechanically unjustified prescription of displacements and traction forces along the boundary, as given by Eqs. (8). This inconsistency leads to the fact that, although both matrices $\mathbf{H}$ and $\mathbf{G}_a$ are singular, they are orthogonal to different bases of vectors:

$$\mathbf{H}^{\mathrm{T}}\mathbf{V} = \mathbf{0} \quad \text{and} \quad \mathbf{G}_a^{\mathrm{T}}\mathbf{Y} = \mathbf{0} \quad \text{but, in general,} \quad \mathbf{V} \neq \mathbf{Y}. \tag{37}$$

Orthogonal properties, as just outlined, will be conveniently explored in the remaining formulations of this paper, namely Eq. (74) and the following developments.

3.2. A spectrally consistent stiffness-type matrix

For the less general case of traction force intensity parameters t_l related with nodal attributes, their number equals the number of degrees of freedom of the problem and matrix $\mathbf{G}$ is square. In this case, errors may be introduced in traction force values at the left and the right sides of a nodal point. In the following, one will make some developments starting from Eq. (31), although the same might be done from Eq. (30) [7]. One might attempt to solve Eq. (31) for the admissible traction parameters $\mathbf{t}$:

$$\mathbf{t} - \mathbf{t}^b = \mathbf{G}_a^{(-1)}\mathbf{H}\left(\mathbf{d} - \mathbf{d}^b\right). \tag{38}$$

An apparent difficulty in obtaining Eq. (38) lies in the fact that $\mathbf{G}_a$ is singular. Fortunately, equation system (31) corresponds mathematically [15] to a problem proposed and solved by Bott and Duffin in 1953 [16]. According to that solution, one proposes the following restricted inverse for $\mathbf{G}_a$,

$$\mathbf{G}_a^{(-1)} = \left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\left(\mathbf{G}_a + \mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}}\right)^{-1} \tag{39}$$

which is more adequate than the standard Bott-Duffin inverse, since it contains a symmetric positive definite but otherwise arbitrary matrix γ which is chosen in order to ensure that the elements of $\mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}}$ and $\mathbf{G}_a$ have approximately the same magnitude, thus avoiding round-off errors during numerical computation. In elastostatics, for instance, the elements of matrix $\mathbf{G}$ are inversely proportional to the shear modulus, which does not affect the orthogonal basis $\mathbf{Z}$. Since $\mathbf{G}_a$ and $\mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}}$ are complementary matrices ($G_a\mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}} \equiv 0$), $G_a + \mathbf{Z}\gamma\mathbf{Z}^{\mathrm{T}}$ is always well conditioned (see Eq. (35) and subsequent considerations). The Bott-Duffin inverse $\mathbf{G}_a^{(-1)}$ in Eq. (39) is a $\{1, 2, 3\}$-inverse of $\mathbf{G}_a$ [15].

Moreover, one may define a vector $\mathbf{p} - \mathbf{p}^b$ of nodal forces that are equivalent in terms of virtual work to the traction force parameters $\mathbf{t} - \mathbf{t}^b$ on the boundary,

$$\mathbf{p} - \mathbf{p}^b = \mathbf{L}^{\mathrm{T}}\left(\mathbf{t} - \mathbf{t}^b\right), \tag{40}$$

where:

$$\mathbf{L}^{\mathrm{T}} \equiv L_{ml} = \int_{\Gamma} u_{im} t_{il} \mathrm{d}\Gamma. \tag{41}$$

Then, it follows from Eqs. (38) and (41) that,

$$\mathbf{p} - \mathbf{p}^b = \mathbf{K}_C\left(\mathbf{d} - \mathbf{d}^b\right) \tag{42}$$

where:

$$\mathbf{K}_C \equiv \mathbf{L}^{\mathrm{T}}\mathbf{G}_a^{(-1)}\mathbf{H} \tag{43}$$

is a stiffness-type matrix obtained in the frame of the conventional boundary element method. There is no reason to believe that this matrix should be any nearer to being symmetric, in general, than the stiffness-type matrix $\mathbf{L}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{H}$. The criticisms expressed in [1] are still valid in case of an admissible matrix $\mathbf{G}_a$. However, matrix $\mathbf{K}_C$, as given in Eq. (43), has improved spectral properties that ensure the equilibrium of the equivalent nodal forces $\mathbf{p}$. This will be demonstrated in the following.

Let the columns of a rectangular matrix $\mathbf{W} \equiv \mathbf{W}_{ns}$ be a basis of the nodal displacements $\mathbf{d}$ related to rigid body displacements. For the moment, one can only say that $\mathbf{W}$ and $\mathbf{Z}$ have the same dimension. For a finite domain, it follows from Eq. (10) that, necessarily,

$$\mathbf{HW} = \mathbf{0} \tag{44}$$

which is a feature related to the physical nature of the fundamental solution. On the other hand, the rigid body displacement functions u_{is}^r may be described along the boundary Γ as a linear combination of the displacement interpolation functions u_{in} and W_{ns},

$$u_{is}^r = u_{im}W_{mt}\omega_{ts}, \tag{45}$$

where $\boldsymbol{\omega} \equiv \omega_{ts}$ is a non-singular square matrix that transforms W_{mt} into the nodal displacements related to u_{is}^r. Pre-multiplying both sides of this equation by t_{il} and integrating, it follows from Eqs. (13) and (41) that,

$$\mathbf{R} = \mathbf{LW}\boldsymbol{\omega}^{\mathrm{T}} \tag{46}$$

and, according to Eq. (25),

$$\mathbf{LW} = \mathbf{Z}\boldsymbol{\lambda}\boldsymbol{\omega}^{-\mathrm{T}} \tag{47}$$

that is, the columns of $\mathbf{LW}$ lie in the space spanned by the rows of $\mathbf{Z}$, to yield:

$$\mathbf{W}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\left(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}}\right) = \boldsymbol{\omega}^{-1}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\left(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}}\right) \equiv \mathbf{0}. \tag{48}$$

Then, given the definitions of $\mathbf{G}_a^{(-1)}$ in Eq. (39) and $\mathbf{K}_C$ in Eq. (42), one obtains from the orthogonality conditions expressed in Eqs. (44) and (48) that $\mathbf{W}^{\mathrm{T}}\mathbf{K}_C = \mathbf{K}_C\mathbf{W} = \mathbf{0}$. As a consequence, the equivalent nodal forces $\mathbf{p}$ of Eq. (42) are always self-equilibrated. Moreover, it may be demonstrated that $\mathrm{rank}(\mathbf{K}_C) = \mathrm{rank}\left(\mathbf{I} - \mathbf{WW}^{\mathrm{T}}\right)$.

Considering Eqs. (32), (48) and (43), one might rewrite Eq. (40) as,

$$\mathbf{p} - \mathbf{p}^b = \mathbf{L}_a^{\mathrm{T}}\left(\mathbf{t} - \mathbf{t}^b\right), \tag{49}$$

where,

$$\mathbf{L}_a \equiv \left(\mathbf{I} - \mathbf{ZZ}^{\mathrm{T}}\right)\mathbf{L} \tag{50}$$

meaning that only the equilibrated (admissible) subsets of vector forces in Eq. (40) are present in the virtual work statement. This restatement is more consistent with the developments of Section 5.

3.3. A simple numerical example

A remarkable or even a slight gain in accuracy cannot be demonstrated with the proposed revisited formulation. The only claim is that it is consistent and not liable to unexpected ill conditioning. The following example [7] illustrates the fact that adding some constant to the fundamental solution does not affect results in the consistent formulation, differently from what occurs in the conventional, inconsistent development. A coarse discretization is chosen to render numerical errors more sensitive to changes of the rigid body constants.

Consider the solution of the Laplace equation on a rectangular domain, shown in Fig. 1. The boundary is discretized with a total of 8 constant elements for both potentials $\mathbf{u}$ and gradients $\mathbf{t}$.

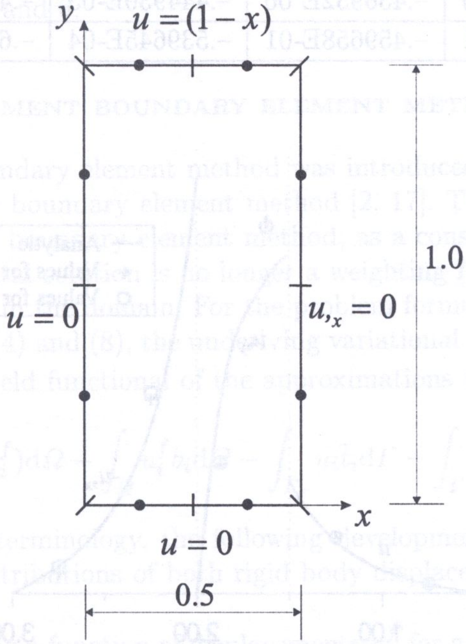

Fig. 1. Rectangular domain and discretization with eight constant elements for the solution of the Laplace equation

The applied boundary conditions are $u = 0$ along the edges $x = 0$ and $y = 0$, $u,_x = 0$ along the edge $x = .5$ and $u = x(1 - x)$ along $y = 1$. The results obtained by considering either $\mathbf{C} = \mathbf{0}$ or the expression of Eq. (34) are represented in Fig. 2 (crosses and circles, respectively), as compared with the analytical solution:

$$u(x, y) = 0.02234116360 \sinh(\pi y) \sin(\pi x)$$
$$+ 0.1542330835\, 10^{-5} \sinh(3\pi y) \sin(3\pi x)$$
$$+ 0.6221263291\, 10^{-9} \sinh(5\pi y) \sin(5\pi x)$$
$$+ 0.1543617478\, 10^{-19} \sinh(2\pi y) \sin(2\pi x) + \ldots$$

The branches of the diagram represent, in this sequence, the gradients on edge $y = 0$, the potentials on edge $x = 0.5$, and the gradients on edges $y = 1$ and $x = 0$ (absolute values are plotted). This example has been repeated for different scales. The results obtained with Eq. (34) presented always the same degree of approximation and were always self-equilibrated. The same behaviour is observed for different boundary elements and discretization meshes. The results of Fig. 2 are defined in Table 1. Two extra columns are added with the results obtained for the dimensions of the problem outlined in Fig. 1 multiplied by 1000. An eight digit accuracy is used in all calculations.

Table 1. Potential and gradient values evaluated at the nodal points for example of Fig. 1

	Analytical values	Values for $\mathbf{C}=0$		Values for $\mathbf{C}$ as in Eq. (34)	
		Sides ×1	Sides ×1000	Sides ×1	Sides ×1000
Gradients	−.268727E-01	−.632023E-02	−.161494E-04	−.383530E-02	−.383503E-05
along $y=0$	−.648386E-01	−.772419E-01	−.943112E-04	−.880434E-01	−.880431E-04
Potentials	.193991E-01	.182500E-01	.198856E-01	.212514E-01	.212514E-01
along $x=5$	.115934E 00	.121802E 00	.123438E 00	.124804E 00	.124804E 00
Gradients	.704797E 00	.833699E 00	.816630E-03	.822898E 00	.822898E-03
along $y=1$	.424514E 00	.291004E 00	.281175E-03	.293489E 00	.293489E-03
Gradients	−.376110E 00	−.436932E 00	−.444930E-03	−.451610E 00	−.451611E-03
along $x=0$	−.610455E-01	−.459658E-01	−.539645E-04	−.606441E-01	−.606441E-04

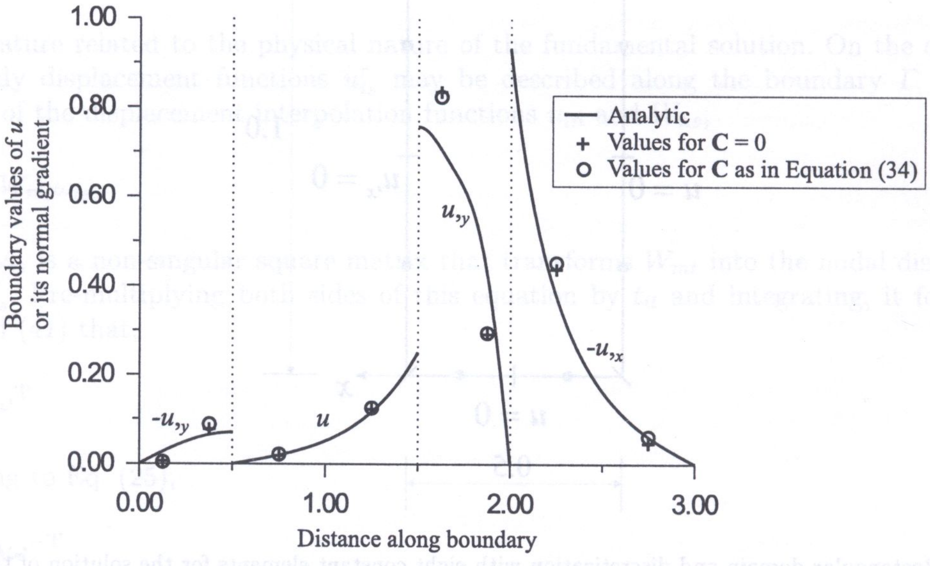

Fig. 2. Results obtained along the boundary for the solution of the Laplace equation: $-u_{,y}$ at $y=0$, u at $x=0.5$, $u_{,y}$ at $y=1$ and $-u_{,x}$ at $x=0$ (see [2])

4. Problem formulation for a variational approach

Consider an elastic body submitted to body forces b_i in the domain Ω and traction forces $\bar{t}_i$ on part Γ_σ of the boundary. Moreover, the displacements $\bar{u}_i$ are known on the complementary part Γ_u of Γ. One is looking for an adequate approximation of the stress field that satisfies equilibrium both in the domain,

$$\sigma_{ji,j} + b_i = 0 \text{ in } \Omega \tag{51}$$

and on the boundary,

$$\sigma_{ji}\eta_j = \bar{t}_i \text{ along } \Gamma_\sigma \tag{52}$$

provided that the following compatibility condition is also satisfied:

$$u_i = \bar{u}_i \text{ on } \Gamma_u. \tag{53}$$

A convenient approximate field solution σ^f_{ij} of the partial differential Eq. (51) may be formulated in terms of a superposition of two types of fields,

$$\sigma^f_{ij} = \sigma^*_{ij} + \sigma^b_{ij} \tag{54}$$

in which σ_{ij}^b is an arbitrary particular solution of Eq. (51),

$$\sigma_{ji,j}^b + b_i = 0 \tag{55}$$

and σ_{ij}^* is expressed as a sum of fundamental solutions, as already introduced in Section 2, Eqs. (1–4).

The displacements corresponding to the field solution σ_{ij}^f are, according to Eq. (1),

$$u_i^f = u_i^* + u_i^b = (u_{im}^* + u_{is}^r C_{sm}) p_m^* + u_i^b \,. \tag{56}$$

Moreover, one may need to interpolate the solution along the boundary, which is done using either the first or both Eqs. (8), depending on the variational principle one is dealing with, as it will be outlined in Sections 5 and 6.

5. The hybrid displacement boundary element method

The hybrid *displacement* boundary element method was introduced by De Figueiredo [5] as an alternative to the hybrid *stress* boundary element method [2, 17]. The main difference between this method and the conventional boundary element method, as a consequence of its variational basis, is the fact that the fundamental solution is no longer a weighting function, but a test function, for expressing the displacements in the domain. For the problem formulated in Section 4 and approximated according to Eqs. (1)–(4) and (8), the underlying variational principle requires the stationary condition [5, 6] of the three-field functional of the approximations (u_i^f, u_i, t_i):

$$\Pi_{HD}(u_i^f, u_i, t_i) = \int_\Omega U_0(u_i^f)\mathrm{d}\Omega - \int_\Omega u_i^f b_i \mathrm{d}\Omega - \int_{\Gamma_\sigma} u_i \bar{t}_i \mathrm{d}\Gamma - \int_\Gamma t_i (u_i^f - u_i)\mathrm{d}\Gamma + \mathrm{Const.} \tag{57}$$

Besides some changes in terminology, the following developments differ conceptually from De Figueiredo's work, as the contributions of both rigid body displacements and body forces are considered properly.

The energy density term, as a function of displacements u_i^f for a linear elastic body, according to Eq. (54), is transformed by means of integration by parts and the application of Green's theorem,

$$\begin{aligned}\int_\Omega U_0(u_i^f)\mathrm{d}\Omega &= \frac{1}{2}\int_\Omega (\sigma_{ij}^* + \sigma_{ij}^b)(u_{i,j}^* + u_{i,j}^b)\mathrm{d}\Omega \\ &= \mathbf{p}^{*\mathrm{T}}\left(\frac{1}{2}\mathbf{F}\mathbf{p}^* + \mathbf{t}^* + \mathbf{b}^*\right) + \mathbf{p}^{*\mathrm{T}}\mathbf{C}^\mathrm{T}\left(\mathbf{R}^\mathrm{T}\mathbf{t}^b + \mathbf{b}^r\right) + \mathrm{Const},\end{aligned} \tag{58}$$

where

$$\mathbf{F} \equiv F_{mn} = \int_\Gamma p_{im}^* u_{in}^* \mathrm{d}\Gamma + u_{mn}^* \tag{59}$$

is a flexibility matrix related to concentrated nodal forces $\mathbf{p}^*$ of the series of fundamental solutions assumed as the homogeneous solution considered in Section 4 (more details are given in Section 6). Note that the rigid body displacements that affect the displacements u_{in}^*, according to Eq. (1), have no influence in the expression of $\mathbf{F}$, as the forces of a fundamental solution are self-equilibrated by definition and produce zero work on rigid body displacements:

$$\int_\Gamma \sigma_{ijm}^* \eta_j u_{is}^r \mathrm{d}\Gamma + \delta_{im} u_{is}^r \equiv 0. \tag{60}$$

Matrix $\mathbf{F}$ is symmetric by definition. Its integral expression involves singularities of the types found in the evaluation of matrices $\mathbf{H}$ and $\mathbf{G}$, except for coefficients about the main diagonal, when indices m and n refer to the same nodal point [2, 17]. These coefficients can only be evaluated in the frame of a spectral property stated below.

In Eq. (58), the term in brackets multiplying the constant $\mathbf{C}$ is null by construction, according to Eq. (21), for the same assumptions of Section 3. However, the explicit consideration of this term will help to simplify the expression of $\Pi_{HD}(u_i^f, u_i, t_i)$ in Eq. (57). The vector of equivalent nodal displacements due to body forces present in Eq. (58) is,

$$\mathbf{t}^* \equiv t_m^* = \int_\Gamma \sigma_{ji}^b \eta_j u_{im}^* \mathrm{d}\Gamma \approx \mathbf{G}\mathbf{t}^b \tag{61}$$

according to Eqs. (17), (19), (20). Vector $\mathbf{b}^*$ is defined in Eq. (14).

As a consequence of these developments, Eq. (58) may be expressed as:

$$\int_\Omega U_0(u_i^f)\mathrm{d}\Omega = \mathbf{p}^{*\mathrm{T}}\left[\frac{1}{2}\mathbf{F}\mathbf{p}^* + \left(\mathbf{G}+\mathbf{C}^\mathrm{T}\mathbf{R}^\mathrm{T}\right)\mathbf{t}^b + \left(\mathbf{b}^* + \mathbf{C}^\mathrm{T}\mathbf{b}^r\right)\right] + \text{Const.} \tag{62}$$

In the term that takes into account the work of the external boundary forces $\bar{t}_i$ in Eq. (57), one may consider either that integration is carried out along Γ instead of Γ_σ, since, after variation, $\delta u_i = 0$ along Γ_u, according to Eq. (53), thus giving rise to the vector $\mathbf{p}$ of nodal forces equivalent to applied tractions $\bar{t}_i$ defined as,

$$\mathbf{d}^\mathrm{T}\mathbf{p} \equiv d_m p_m = d_m \int_\Gamma u_{im}\bar{t}_i \mathrm{d}\Gamma \tag{63}$$

or, alternatively, that $\mathbf{p}$ is in part a set of nodal forces equivalent to known surface forces $\bar{t}_i$ along part Γ_σ of the boundary,

$$\mathbf{d}^\mathrm{T}\mathbf{p} \equiv d_m p_m = d_m \int_{\Gamma_\sigma} u_{im}\bar{t}_i \mathrm{d}\Gamma \tag{64}$$

and in part a set of unknowns corresponding to reaction forces along the complementary boundary segment Γ_u. Both interpretations are conceptually valid.

Substituting in Eq. (57) the functions u_i^f, u_i, t_i for their values given in Eqs. (56) and (8), and considering Eqs. (41), (62) and also the approximation of u_i^b given by Eq. (19), one arrives at the matrix expression of the functional:

$$\Pi_{HD}(u_i^f, u_i, t_i) = \mathbf{p}^{*\mathrm{T}}\left[\frac{1}{2}\mathbf{F}\mathbf{p}^* - \left(\mathbf{G}+\mathbf{C}^\mathrm{T}\mathbf{R}^\mathrm{T}\right)\left(\mathbf{t}-\mathbf{t}^b\right)\right] - \mathbf{d}^\mathrm{T}\mathbf{p} + \mathbf{t}^\mathrm{T}\mathbf{L}\left(\mathbf{d}-\mathbf{d}^b\right) + \text{Const.} \tag{65}$$

Observe that $\mathbf{b}^*$ cancels out. In order to transform this functional into a more useful form, define the equivalent nodal forces $\mathbf{p}^b$ related to the traction forces $\mathbf{t}^b \equiv t_l^b \equiv \sigma_{jl}^b \eta_j$,

$$\mathbf{p}^b \equiv \mathbf{L}^\mathrm{T}\mathbf{t}^b \tag{66}$$

and add the expression $\mathbf{d}^\mathrm{T}\left(\mathbf{p}^b - \mathbf{L}^\mathrm{T}\mathbf{t}^b\right) \equiv \mathbf{0}$, a constant, to Eq. (65). Then, Eq. (65) becomes, after some rearrangement,

$$\Pi_{HD}\left(u_i^f, u_i, t_i\right) = \mathbf{p}^{*\mathrm{T}}\left[\frac{1}{2}\mathbf{F}\mathbf{p}^* - \left(\mathbf{G}+\mathbf{C}^\mathrm{T}\mathbf{R}^\mathrm{T}\right)\left(\mathbf{t}-\mathbf{t}^b\right)\right] - \mathbf{d}^\mathrm{T}\left(\mathbf{p}-\mathbf{p}^b\right) + \left(\mathbf{t}-\mathbf{t}^b\right)^\mathrm{T}\mathbf{L}\left(\mathbf{d}-\mathbf{d}^b\right) + \text{Const.} \tag{67}$$

Recalling the developments of Section 2, one concludes that the functional above should be valid only for balanced traction forces $\left(\mathbf{t}-\mathbf{t}^b\right) \equiv \left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^\mathrm{T}\right)\left(\mathbf{t}-\mathbf{t}^b\right)$. As a consequence, one finally expresses $\Pi_{HD}\left(u_i^f, u_i, t_i\right)$ as,

$$\begin{aligned}\Pi_{HD}\left(u_i^f, u_i, t_i\right) = \mathbf{p}^{*\mathrm{T}}\left[\frac{1}{2}\mathbf{F}\mathbf{p}^* - \mathbf{G}_a\left(\mathbf{t}-\mathbf{t}^b\right)\right] \\ - \mathbf{d}^{\mathrm{T}}\left(\mathbf{p}-\mathbf{p}^b\right) + \left(\mathbf{t}-\mathbf{t}^b\right)^{\mathrm{T}}\mathbf{L}_a\left(\mathbf{d}-\mathbf{d}^b\right) + \text{Const}\,,\end{aligned} \tag{68}$$

where $\mathbf{G}_a$ and $\mathbf{L}_a$ are given by Eqs. (32) and (50) respectively.

Then, applying the fundamental lemma of the variational calculus, one arrives at the three sets of matrix equations:

$$\mathbf{F}\mathbf{p}^* = \mathbf{G}_a\left(\mathbf{t}-\mathbf{t}^b\right), \tag{69}$$

$$\mathbf{p}-\mathbf{p}^b = \mathbf{L}_a^{\mathrm{T}}\left(\mathbf{t}-\mathbf{t}^b\right), \tag{70}$$

$$\mathbf{G}_a^{\mathrm{T}}\mathbf{p}^* = \mathbf{L}_a\left(\mathbf{d}-\mathbf{d}^b\right). \tag{71}$$

Equations (70) and (71) are consistent by construction, as,

$$\mathbf{W}^{\mathrm{T}}\left(\mathbf{p}-\mathbf{p}^b\right) = \mathbf{W}^{\mathrm{T}}\mathbf{L}_a^{\mathrm{T}}\left(\mathbf{t}-\mathbf{t}^b\right) = \mathbf{0}, \tag{72}$$

$$\mathbf{Z}^{\mathrm{T}}\mathbf{G}_a^{\mathrm{T}}\mathbf{p}^* = \mathbf{Z}^{\mathrm{T}}\mathbf{L}_a\left(\mathbf{d}-\mathbf{d}^b\right) = \mathbf{0} \tag{73}$$

according to the definitions of $\mathbf{L}_a$ and $\mathbf{G}_a$, and also considering Eq. (48).

To investigate the consistency of Eq. (69), one observes first that, if the admissible matrix $\mathbf{G}_a$ is by construction orthogonal to $\mathbf{Z}$, according to Eq. (32), there also exists an orthonormal basis $\mathbf{Y}$ such that

$$\mathbf{G}_a^{\mathrm{T}}\mathbf{Y} = \mathbf{0}. \tag{74}$$

Then, the admissible set of singular forces $\mathbf{p}^*$ that can be transformed into displacements in Eq. (69) must necessarily be orthogonal to $\mathbf{Y}$, coherently with Eq. (71):

$$\mathbf{Y}^{\mathrm{T}}\mathbf{p}^* = \mathbf{0}. \tag{75}$$

As a consequence, if matrix $\mathbf{F}$ in Eq. (69) is singular, as proposed by Dumont [2, 17], then it must also be orthogonal to $\mathbf{Y}$, as followed by De Figueiredo [5]:

$$\mathbf{F}\mathbf{Y} = \mathbf{0}. \tag{76}$$

This is the criterion needed for the determination of the coefficients about the main diagonal of matrix $\mathbf{F}$, in the hybrid displacement boundary element method (see more considerations on this feature in Section 5).

According to Eq. (76), Eq. (69) is equivalent to,

$$\mathbf{t}-\mathbf{t}^b = \mathbf{G}_a^{-1}\left(\mathbf{I}-\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)\mathbf{F}\mathbf{p}^* \tag{77}$$

recalling Eq. (39), which defines $\mathbf{G}_a^{-1}$ as a $\{1,2,3\}$ – inverse of $\mathbf{G}_a$. On the other hand, it may be shown that the matrix expression $\mathbf{G}_a^{-1}\left(\mathbf{I}-\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)$ is mathematically equivalent to the least-squares inverse $(\mathbf{G}_a)_{LS}^{-1}$ of $\mathbf{G}_a$:

$$(\mathbf{G}_a)_{LS}^{-1} = \left(\mathbf{G}_a^{\mathrm{T}}\mathbf{G}_a + \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)^{-1}\mathbf{G}_a^{\mathrm{T}} \equiv \mathbf{G}_a^{-1}\left(\mathbf{I}-\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right). \tag{78}$$

This matrix $(\mathbf{G}_a)_{LS}^{-1}$ is a $\{1,2,3,4\}$ – inverse of $\mathbf{G}_a$. Also, it may be shown that:

$$\left(\mathbf{G}_a^{\mathrm{T}}\right)_{LS}^{-1} = \left(\mathbf{G}_a\mathbf{G}_a^{\mathrm{T}} + \mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)^{-1}\mathbf{G}_a \equiv \left((\mathbf{G}_a)_{LS}^{-1}\right)^{\mathrm{T}} \tag{79}$$

In Eq. (71), one is interested only in the admissible subset of the vector $\mathbf{p}^*$, as defined in Eq. (75). Then, one obtains from Eq. (71) that, in principle,

$$\mathbf{p}^* = \left(\mathbf{G}_a^{\mathrm{T}}\right)^{-1}\mathbf{L}_a\left(\mathbf{d}-\mathbf{d}^b\right), \tag{80}$$

where

$$\left(\mathbf{G}_a^{\mathrm{T}}\right)^{-1} = \left(\mathbf{I}-\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)\left(\mathbf{G}_a^{\mathrm{T}} + \mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)^{-1} \tag{81}$$

is the Bott–Duffin inverse, therefore a $\{1,2,3\}$ – inverse, of matrix $\mathbf{G}_a^{\mathrm{T}}$, where, in general:

$$\left(\mathbf{G}_a^{\mathrm{T}}\right)^{-1} \neq \left(\mathbf{G}_a^{-1}\right)^{\mathrm{T}}. \tag{82}$$

The use of Eq. (80) is correct, although unnecessary. In fact, it may be shown that,

$$\left(\mathbf{G}_a^{\mathrm{T}}\right)^{-1}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right) \equiv \left(\mathbf{G}_a^{\mathrm{T}}\right)_{LS}^{-1}, \tag{83}$$

where $\left(\mathbf{G}_a^{\mathrm{T}}\right)_{LS}^{-1}$ is the least-squares inverse introduced in Eq. (79). Then, considering Eqs. (78) and (79), one obtains,

$$\left(\mathbf{G}_a^{\mathrm{T}}\right)^{-1}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right) = \left(\mathbf{I}-\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)\left(\mathbf{G}_a^{-1}\right)^{\mathrm{T}} \tag{84}$$

and an alternative form of Eq. (80) in the equivalent, more convenient way,

$$\mathbf{p}^* = \left(\mathbf{I}-\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right)\left(\mathbf{G}_a^{-1}\right)^{\mathrm{T}}\mathbf{L}\mathbf{d} \tag{85}$$

taking advantage of the fact that matrix $\mathbf{G}_a^{-1}$ has already been evaluated, as it is required in Eq. (77).

Finally, substituting for $\mathbf{p}^*$, as given in Eq. (85), in Eq. (69) and considering Eqs. (70) and (77), one arrives at a stiffness relation between nodal displacements and equivalent nodal forces,

$$\mathbf{K}_D\left(\mathbf{d}-\mathbf{d}^b\right) = \mathbf{p}-\mathbf{p}^b, \tag{86}$$

where,

$$\mathbf{K}_D = \mathbf{L}_a^{\mathrm{T}}\mathbf{G}_a^{-1}\mathbf{F}\left(\mathbf{G}_a^{-1}\right)^{\mathrm{T}}\mathbf{L}_a \equiv \mathbf{L}^{\mathrm{T}}\mathbf{G}_a^{-1}\mathbf{F}\left(\mathbf{G}_a^{-1}\right)^{\mathrm{T}}\mathbf{L} \tag{87}$$

is a stiffness matrix. According to Eq. (48), $\mathbf{K}_D$ is by construction orthogonal to rigid body displacements, independently from the properties of the matrix $\mathbf{F}$.

De Figueiredo [5] introduced the hybrid displacement boundary element method with no consideration of the rigid body displacements that are inherent to a fundamental solution, Eq. (1). Then, $\mathbf{C}=\mathbf{0}$ in Eq. (65) and matrices $\mathbf{G}$ and $\mathbf{L}$ replace $\mathbf{G}_a$ and $\mathbf{L}_a$ in Eqs. (69)–(71):

$$\mathbf{F}\mathbf{p}^* = \mathbf{G}\mathbf{t} + \mathbf{b}^*, \tag{88}$$

$$\mathbf{p} = \mathbf{L}^{\mathrm{T}}\mathbf{t}, \tag{89}$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{p}^* = \mathbf{L}\mathbf{d}. \tag{90}$$

Moreover, it was assumed that, instead of Eq. (76),

$$\mathbf{F}\widetilde{\mathbf{Y}} = \mathbf{0}, \tag{91}$$

where $\widetilde{\mathbf{Y}}$ is the solution of the inconsistent version of Eq. (71) for rigid body displacements:

$$\mathbf{G}^{\mathrm{T}}\widetilde{\mathbf{Y}} = \mathbf{L}\mathbf{W}. \tag{92}$$

Then, after evaluation of the diagonal elements of $\mathbf{F}$, according to Eq. (91), one arrives at:

$$\mathbf{K}_D\mathbf{d} = \mathbf{p} + \mathbf{L}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{b}^* \qquad \text{with} \qquad \mathbf{K}_D = \mathbf{L}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{F}\left(\mathbf{G}^{-1}\right)^{\mathrm{T}}\mathbf{L}. \tag{93}$$

To assess the coherence of this formulation, consider Eq. (74) written as:

$$\left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\mathbf{G}^{\mathrm{T}}\mathbf{Y} = \mathbf{0}. \tag{94}$$

As a consequence,

$$\mathbf{G}^{\mathrm{T}}\mathbf{Y} = \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{Y} \equiv \widetilde{\mathbf{Z}} \tag{95}$$

in which $\widetilde{\mathbf{Z}}$ is a non-orthonormal basis of the same space spanned by $\mathbf{Z}$ (since $\mathbf{Z}\mathbf{Z}^{\mathrm{T}}$ is an orthogonal projector). Now, comparing Eqs. (92) and (95), and considering Eq. (47), one concludes that $\widetilde{\mathbf{Y}}$ in Eqs. (91) and (92) is a non-orthonormal basis of the space spanned by $\mathbf{Y}$. As a consequence, Eqs. (76) and (91) are equivalent. One arrives at the same conclusion if one pre-multiplies both sides of Eq. (92) by $\left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)$, thus obtaining, as a consequence of the orthogonality expressed by Eq. (48):

$$\left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\mathbf{G}^{\mathrm{T}}\widetilde{\mathbf{Y}} = \left(\mathbf{I} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\right)\mathbf{L}\mathbf{W} = \mathbf{0}. \tag{96}$$

Comparing this equation with Eq. (94), one readily sees that $\widetilde{\mathbf{Y}}$ and $\mathbf{Y}$ span the same space. Moreover, matrix $\mathbf{K}_D$, as indicated in Eqs. (86) and (93), is one and the same matrix, provided that $\mathbf{G}$ may be inverted (is not ill conditioned).

The brief outline of this section is an important theoretical contribution to the hybrid displacement boundary element method, since it assesses ant attests the spectral consistency of the stiffness matrix $\mathbf{K}_D$, obtained by De Figueiredo [5]. However, the vectors equivalent to body forces should be expressed as in Eq. (86), and not as in Eqs. (88) and (93). In fact, the approach presented by De Figueiredo for body forces is incorrect, since the energy density $U_0\left(u_i^f\right)$ is expressed as a function of u_i^*, and not of u_i^f, as outlined in this Section.

Once Eqs. (69)–(71), possibly combined as Eq. (86), are solved for some well-posed, but otherwise general, problem, the vector of concentrated forces $\mathbf{p}^*$ is known and results at internal points are evaluated using Eq. (56) with constants C_{sm} obtained by Eq. (34).

5.1. Numerical example: potential problem in an infinite medium

This example deals with the solution of the Helmholtz equation for a cavity in an infinite two-dimensional region [6]. See [18] for the general, frequency-dependent formulation of the hybrid displacement boundary element method. The extension of the formulation for unbounded regions is briefly outlined in Section 7, in a wider context. The cavity, as shown in Fig. 3, is discretized with 20 linear boundary elements. For a source located at point (4, 1) that propagates a potential given by the fundamental solution

$$\theta^* = \frac{-1}{2\pi}\ln(r) + \frac{-1}{2\pi}\left(\frac{\pi}{2}\mathrm{Y}_0(kr) - \ln(r) - \left(\ln\left(\frac{k}{2}\right) + \gamma\right)\mathrm{J}_0(kr)\right) \tag{97}$$

one evaluates a vector $\mathbf{p}$ of equivalent nodal fluxes, according to Eq. (70). The response of the problem is given in Table 2 for a frequency number $k = 0.3$. Both numerical and analytical values obtained at points along the dashed line shown in Fig. 3 are given, as well as at some boundary points.

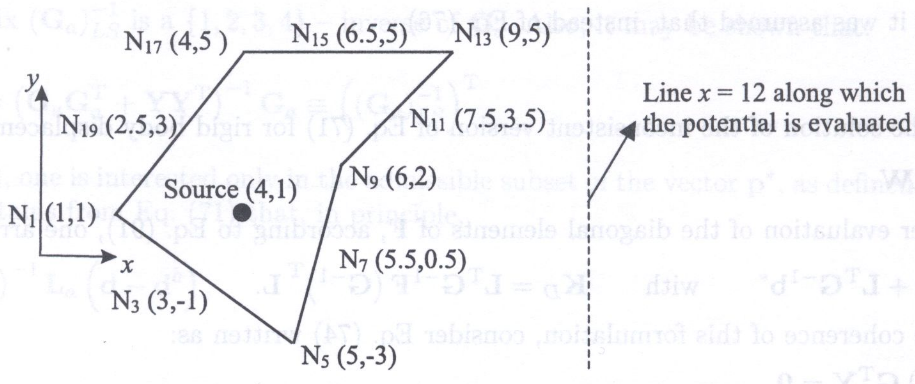

Fig. 3. Potential problem — irregular shaped cavity in an infinite domain discretized with 20 linear boundary elements

Table 2. Potential problem in a cavity – Potential values at points indicated in Fig. 3

Point in Ω	Potential		Point along Γ	Potential	
(x, y)	Analytic	HBEM		Analytic	HBEM
(12, 6)	−0.06551	−0.06521	N1	−0.17104	−0.16774
(12, 5)	−0.08769	−0.08738	N3	−0.13123	−0.13351
(12, 4)	−0.10533	−0.10502	N5	−0.19971	−0.19713
(12, 3)	−0.11801	−0.11769	N7	−0.07764	−0.09703
(12, 2)	−0.12560	−0.12525	N9	−0.13123	−0.14155
(12, 1)	−0.12813	−0.12773	N11	−0.20162	−0.20231
(12, 0)	−0.12560	−0.12516	N13	−0.18168	−0.18113
(12, −1)	−0.11801	−0.11753	N15	−0.20362	−0.20278
(12, −2)	−0.10533	−0.10485	N17	−0.19801	−0.19581
(12, −3)	−0.08769	−0.08726	N19	−0.14726	−0.14919

6. An outline of the hybrid stress boundary element method

The hybrid stress boundary element method is based on the Hellinger–Reissner potential,

$$-\Pi_R(\sigma^f_{ij}, u_i) = \int_\Omega \left[U_0^C\left(\sigma^f_{ij}\right) + \left(\sigma^f_{ji},_j + b_i\right) u_i\right] \mathrm{d}\Omega - \int_\Gamma \sigma^f_{ji}\eta_j u_i \mathrm{d}\Gamma + \int_{\Gamma_\sigma} u_i \bar{t}_i \mathrm{d}\Gamma + \text{Const} \tag{98}$$

as first applied by T. H. H. Pian [3] to finite elements. In 1987, the present author generalized Pian's ideas for considering the stress field in the domain as a series of fundamental, singular solutions σ^f_{ij}, according to Eq. (54), thus arriving at a boundary integral formulation [17].

The complementary energy density $U_0^C\left(\sigma^f_{ij}\right)$ in Eq. (98) is a function of the stress field and yields Eq. (99),

$$\int_\Omega U_0^C\left(\sigma^f_{ij}\right) \mathrm{d}\Omega = \mathbf{p}^{*\mathrm{T}}\left[\frac{1}{2}\mathbf{F}\mathbf{p}^* + \mathbf{b}^b\right] + \text{Const} \tag{99}$$

after integration by parts and the application of Green's theorem, with the same flexibility matrix $\mathbf{F}$ introduced in Eq. (59) and vector $\mathbf{b}$ of equivalent nodal displacements due to the applied body forces:

$$\mathbf{b}^b \equiv b^b_m = \int_\Gamma p^*_{im} u^b_i \mathrm{d}\Gamma + \delta_{im} u^b_i. \tag{100}$$

It is observed that no rigid body constants appear in Eqs. (98) to (100), as a consequence of the fact that a fundamental solution is always self equilibrated, according to Eq. (60). Considering the approximation of u_i^b given by Eq. (19), one may express $\mathbf{b}^b$ as,

$$\mathbf{b}^b = \mathbf{H}\mathbf{d}^b \tag{101}$$

similarly to the argument from Eq. (17) to Eq. (20), and rewrite Eq. (99) in a way more convenient to the further developments:

$$\int_\Omega U_0^C\left(\sigma_{ij}^f\right) \mathrm{d}\Omega = \mathbf{p}^{*\mathrm{T}}\left[\frac{1}{2}\mathbf{F}\mathbf{p}^* + \mathbf{H}\mathbf{d}^b\right] + \text{Const}\,. \tag{102}$$

Although already defined, the flexibility matrix $\mathbf{F} \equiv F_{mn}$ and the kinematic transformation matrix $\mathbf{H} \equiv H_{mn}$ are given again in a compact notation as:

$$[\mathbf{F}\ \mathbf{H}] = \int_\Gamma \sigma_{ijm}^* \eta_j \langle u_{in}^*\ u_{in}\rangle \,\mathrm{d}\Gamma + \delta_{im}\langle u_{in}^*\ u_{in}\rangle\,. \tag{103}$$

Owing to the singularity of the fundamental solution, the boundary integral represented by equations above is singular and has to be split into a Cauchy principal value and a discontinuous term. Related to this singularity, a generalized Kronecker delta is introduced, according to Eq. (3). Coefficients about the main diagonal of the flexibility matrix $\mathbf{F}$, for m and n referring to the same node, cannot be evaluated by means of this integral, since singularities of the type $\ln(r)/r$, for two-dimensional problems, or r^{-3}, for three-dimensional problems, arise as $r \to 0$. This mathematical impossibility is consistent with the assumption — common to all boundary element formulations – that the nodal point is situated outside the domain Ω, although infinitely close to it, which means that the corresponding equivalent nodal displacements F_{mn} are undetermined in terms of virtual work. The determination of these coefficients has to be carried out indirectly by requiring that $\mathbf{F}$ satisfies some orthogonality criterion, as given in Eq. (112) below, for the present formulation.

The kinematic transformation matrix $\mathbf{H}$ introduced in this Section is the same double-layer potential matrix that arises in the conventional, collocation boundary element method, as given in Section 3. Its evaluation, according to Eq. (103), should be considered a standard procedure. For the sake of a better understanding of the evaluation of the flexibility matrix $\mathbf{F}$, however, it is advisable to express Eq. (103) as,

$$\begin{aligned}[\mathbf{F}\ \mathbf{H}] &\equiv [\mathbf{F}_{\mathrm{fp}}\ \mathbf{H}_{\mathrm{fp}}] + [\mathbf{F}_{\mathrm{disc}}\ \mathbf{H}_{\mathrm{disc}}] \\ &= fp\int_\Gamma \sigma_{ijm}^*\eta_j\langle u_{in}^*\ u_{in}\rangle\,\mathrm{d}\Gamma + \left(\int_{\Gamma_0}\sigma_{ijm}^*\eta_j\langle u_{in}^*\ u_{in}\rangle\,\mathrm{d}\Gamma + \delta_{im}\langle u_{in}^*\ u_{in}\rangle\right)\end{aligned} \tag{104}$$

in which $()_{\mathrm{fp}}$ is a finite-part integral and $()_{\mathrm{disc}}$ comprises the discontinuous terms of the general, singular integrals of Eq. (103). When m and n refer to different nodal points, there are no singularities involved, which means that $\mathbf{H}_{\mathrm{disc}}$ is a block-diagonal matrix, as it is well established in the literature. The block-diagonal submatrices of $\mathbf{H}$, corresponding to $\mathbf{H}_{\mathrm{disc}}$ added to the block-diagonal submatrices of $\mathbf{H}_{\mathrm{fp}}$, may be obtained indirectly using the orthogonality of $\mathbf{H}$ to rigid body nodal displacements, as expressed in Eq. (44). However, the explicit expression of $\mathbf{H}_{\mathrm{disc}}$, as given in Eq. (104), is needed for the complete evaluation of $\mathbf{F}$.

In fact, observing that the interpolation function u_{in} introduced in Eq. (8) and used in Eq. (103) is by definition equal to unity when n and m refer to the same nodal point, Eq. (104) may be expressed as,

$$[\mathbf{F}\ \mathbf{H}] \equiv [\mathbf{F}_{\mathrm{fp}}\ \mathbf{H}_{\mathrm{fp}}] + [\mathbf{F}_{\mathrm{disc}}\ \mathbf{H}_{\mathrm{disc}}] = [\mathbf{F}_{\mathrm{fp}}\ \mathbf{H}_{\mathrm{fp}}] + \mathbf{H}_{\mathrm{disc}}[\mathbf{U}^*\ \mathbf{I}]\,, \tag{105}$$

where $\mathbf{U}^* \equiv U_{mn}^*$ is a square matrix given as the fundamental solution u_{im}^* measured at the degree of freedom n for a unitary, concentrated force p_m^* applied at the degree of freedom m. The evaluation of the coefficients in $\mathbf{F}_{\text{fp}}$ involves the same mathematical considerations as in the evaluation of $\mathbf{H}_{\text{fp}}$ in Eq. (104), added to improper-integral considerations related to u^*_{im}, as occurs in the evaluation of the single-layer potential matrix $\mathbf{G}$ of the conventional, collocation boundary element method, according to Eq. (12). It is observed that the block-diagonal coefficients of $\mathbf{F}$ in Eq. (105) cannot be evaluated directly, since U^*_{mn} is undefined for n and m referring to the same nodal point — refer to Eq. (112) below. More considerations on U^*_{mn} are given in [19].

Returning to the evaluation of the Hellinger-Reissner potential, Eq. (98), one may make the same considerations for the term that takes into account the work of the external boundary forces $\bar{t}_i$ as in Section 5, according to either Eq. (63) or (64). Substituting in Eq. (98) the function σ^f_{ij} for its expression according to Eqs. (54) and (4), the function u_i according to Eq. (8) and considering Eq. (102), one arrives at the matrix expression of the functional:

$$-\delta\Pi_R = \delta\mathbf{p}^{*\mathrm{T}}\left[\mathbf{F}\mathbf{p}^* - \mathbf{H}\left(\mathbf{d}-\mathbf{d}^b\right)\right] + \delta\mathbf{d}^{\mathrm{T}}\left[\mathbf{p}-\mathbf{p}^b-\mathbf{H}^{\mathrm{T}}\mathbf{p}^*\right] = \mathbf{0}. \tag{106}$$

The vector $\mathbf{p}^b$ are nodal forces equivalent to applied body forces:

$$\mathbf{p}^b \equiv p^b_m = \int_\Gamma \sigma^b_{ji}\eta_j u_{im}\,\mathrm{d}\Gamma. \tag{107}$$

It is observed that this expression can be used directly to obtain $\mathbf{p}^b$ in Eq. (66).

For arbitrary variations $\delta\mathbf{p}^*$ and $\delta\mathbf{d}$, two sets of equations originate from Eq. (106):

$$\begin{aligned}\mathbf{F}\mathbf{p}^* &= \mathbf{H}\left(\mathbf{d}-\mathbf{d}^b\right),\\ \mathbf{H}^{\mathrm{T}}\mathbf{p}^* &= \mathbf{p}-\mathbf{p}^b.\end{aligned} \tag{108}$$

For a finite domain, $\mathbf{H}$ is singular by its construction, as stated in Eq. (44). As a consequence, there is an orthogonal basis $\mathbf{V}$ such that:

$$\mathbf{H}^{\mathrm{T}}\mathbf{V} = \mathbf{0}. \tag{109}$$

Moreover, it may be verified that, in the second of Eqs. (108):

$$\mathbf{W}^{\mathrm{T}}\left(\mathbf{p}-\mathbf{p}^b\right) = \mathbf{0}. \tag{110}$$

Then, one must have, for physical consistency,

$$\mathbf{V}^{\mathrm{T}}\mathbf{p}^* = \mathbf{0} \tag{111}$$

from which follows, in the first of Eqs. (108), that,

$$\mathbf{F}\mathbf{V} = \mathbf{0} \tag{112}$$

if $\mathbf{F}$ is singular. This equation is the key for the evaluation of the coefficients about the main diagonal of $\mathbf{F}$, which cannot be directly obtained by integration.

Considering the spectral properties given by Eqs. (111) and (112), one may solve the first of Eqs. (108) for $\mathbf{p}^*$, in terms of generalized inverses [15] and introduce its expression into the second of Eqs. (108), thus arriving at the relation,

$$\mathbf{H}^{\mathrm{T}}\left(\mathbf{F}+\mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{H}\mathbf{d} = \mathbf{p} - \mathbf{t}^b + \mathbf{H}^{\mathrm{T}}\left(\mathbf{F}+\mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{b}^b, \tag{113}$$

where $\mathbf{K}_S = \mathbf{H}^{\mathrm{T}}\left(\mathbf{F}+\mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{H}$ is a symmetric, positive semi-definite stiffness matrix. Owing to the spectral property of $\mathbf{H}$ given by Eq. (44), this stiffness matrix is by its construction orthogonal to the rigid body displacements.

Interested readers are referred to some of the articles written by the author in the last decade for a more detailed description of the hybrid stress boundary element method.

6.1. Numerical example: application to linear elastic fracture mechanics

A simple illustration of the applicability of the method outlined in this Section is shown in Fig. 4. It represents a convergence study on the evaluation of stress intensity factors at the tip of a skew edge crack in the rectangular plate shown. Stress intensity factors K_I and K_{II} are given in the vertical axis, for an increasing number of linear elements used to discretize the crack [8].

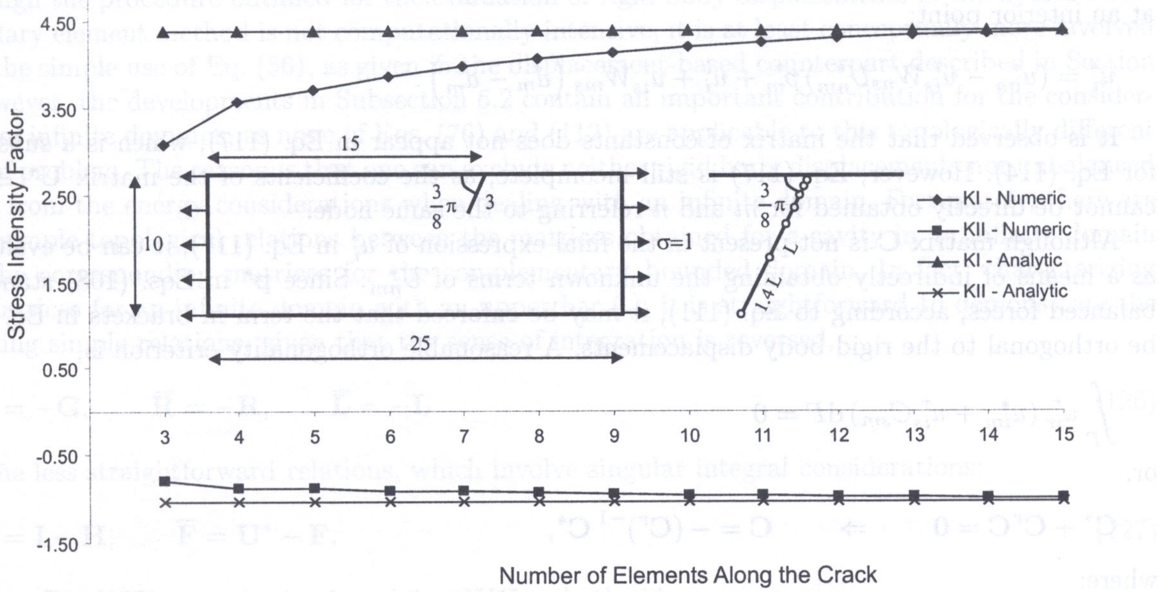

Fig. 4. Skew edge crack in a rectangular plate subjected to uniaxial loading, for element lengths varying with a 1:1.4 ratio [8]

6.2. Evaluation of displacements in the domain

In the hybrid displacement boundary element method of Section 5, the displacements at interior points are given directly from Eq. (56). This means that the contribution of the rigid body displacements is implicitly considered in the formulation.

In the hybrid stress boundary element method, on the other hand, the rigid body contribution has to be evaluated explicitly during post-analysis, since no reference is previously made to the constants C_{sm}. Although the procedure outlined may seem quite cumbersome, the consequences of the present developments are paramount for the extension of both versions of the hybrid boundary element methods to infinite domains. Equation (56) is rewritten in an equivalent form, in which the rigid body displacements are expressed separately, for some vector of parameters $\mathbf{r} \equiv r_s$:

$$u_i^f = u_i^* + u_i^b = (u_{im}^* + u_{is}^r C_{sm})\, p_m^* + u_i^b + u_{is}^r r_s. \tag{114}$$

First of all, consider, for the sake of simplicity, that u_{is}^r in Eq. (114) is so normalized as to yield the orthonormal basis $\mathbf{W}$ of rigid body displacements when evaluated at the nodal points, meaning that ω in Eq. (45) is an identity matrix.

To be consistent, Eq. (114) must be valid at the nodal points, thus yielding,

$$\mathbf{d} = (\mathbf{U}^* + \mathbf{WC})\,\mathbf{p}^* + \mathbf{d}^b + \mathbf{Wr} \quad \text{or} \quad d_m = (U_{mn}^* + W_{ms} C_{sn}) p_n^* + d_m^b + W_{ms} r_s \tag{115}$$

in matrix and index notation, respectively. In this equation, $\mathbf{U}^*$ is a symmetric matrix obtained by expressing the fundamental solution u_{in}^* at the nodal points, as already introduced in Eq. (105).

When m and n refer to the same nodal point, the corresponding coefficients can only be evaluated by means of a spectral property, as explained in the following. However, supposing that $\mathbf{U}^*$ is completely known, one obtains the rigid body parameters $\mathbf{r}$ by pre-multiplying both sides of Eq. (115) by $\mathbf{W}^{\mathrm{T}}$ (recalling that $\mathbf{W}$ is orthonormal):

$$\mathbf{r} = \mathbf{W}^{\mathrm{T}}\left(\mathbf{d} - \mathbf{d}^b\right) - (\mathbf{W}^{\mathrm{T}}\mathbf{U}^* + \mathbf{C})\,\mathbf{p}^* \quad \text{or} \quad r_s = W_{ms}(d_m - d_m^b) - (W_{ms}U^*_{mn} + C_{sn})p^*_n. \tag{116}$$

Now, substituting this expression into Eq. (114), one obtains the final expression of displacements at an interior point:

$$u_i^* = (u^*_{im} - u^r_{is}W_{ns}U^*_{nm})\,p^*_m + u^b_i + u^r_{is}W_{ms}\left(d_m - d^b_m\right). \tag{117}$$

It is observed that the matrix of constants does not appear in Eq. (117), which is a substitute for Eq. (114). However, Eq. (117) is still incomplete, as the coefficients of the matrix $\mathbf{U}^* \equiv U^*_{nm}$ cannot be directly obtained for m and n referring to the same node.

Although matrix $\mathbf{C}$ is not present in the final expression of u_i^* in Eq. (117), it can be evaluated, as a means of indirectly obtaining the unknown terms of U^*_{nm}. Since $\mathbf{p}^*$ in Eqs. (108) stands for balanced forces, according to Eq. (111), it may be enforced that the term in brackets in Eq. (114) be orthogonal to the rigid body displacements. A reasonable orthogonality criterion is,

$$\int_\Gamma u^r_{ir}\,(u^*_{im} + u^r_{is}C_{sm})\,\mathrm{d}\Gamma = 0 \tag{118}$$

or,

$$\mathbf{C}^* + \mathbf{C}^r\mathbf{C} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{C} = -\left(\mathbf{C}^r\right)^{-1}\mathbf{C}^*, \tag{119}$$

where:

$$\mathbf{C}^r \equiv C^r_{rs} = \int_\Gamma u^r_{ir}u^r_{is}\mathrm{d}\Gamma \quad \text{and} \quad \mathbf{C}^* \equiv C^*_{rm} = \int_\Gamma u^r_{ir}u^*_{im}\mathrm{d}\Gamma. \tag{120}$$

Now, since the term in brackets in Eq. (114) is required to be orthogonal to rigid body displacements, the orthogonality criterion (to unbalanced forces),

$$(\mathbf{U}^* + \mathbf{WC})\,\mathbf{V} = \mathbf{0} \tag{121}$$

must hold, according to Eqs. (109), (111) and (112). Then, Eq. (121) is the criterion needed for evaluating the coefficients of $\mathbf{U}^*$, when m and n refer to the same node [19].

Once $\mathbf{U}^*$ is completely determined, absolute displacement results at internal points can be evaluated according to Eq. (117), which is equivalent to Eq. (114) for $\mathbf{C} \equiv C_{sm}$ given by Eq. (119).

It is worth investigating the conceptual difference between the matrix of constants $\mathbf{C}$ obtained according to Eq. (119) and the one used in Sections 3 and 4. For this sake, consider the rigid body displacements u^r_{is} expressed along the boundary Γ in terms of the traction force interpolation functions t_{il} of Eq. (8) and some multipliers $\widetilde{\mathbf{W}} \equiv \widetilde{W}_{ls}$:

$$u^r_{is} = t_{il}\widetilde{W}_{ls}. \tag{122}$$

Then, one may write, from Eqs. (120),

$$\mathbf{C}^r \equiv C^r_{rs} = \int_\Gamma u^r_{ir}u^r_{is}\mathrm{d}\Gamma = \widetilde{W}_{lr}\int_\Gamma t_{il}u^r_{is}\mathrm{d}\Gamma = \widetilde{\mathbf{W}}^{\mathrm{T}}\mathbf{R}, \tag{123}$$

$$\mathbf{C}^* \equiv C^*_{rm} = \int_\Gamma u^r_{ir}u^*_{im}\mathrm{d}\Gamma = \widetilde{W}_{lr}\int_\Gamma t_{il}u^*_{im}\mathrm{d}\Gamma = \widetilde{\mathbf{W}}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} \tag{124}$$

and Eq. (119) becomes, in matrix form,

$$(\mathbf{G} + \mathbf{C}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}})\,\widetilde{\mathbf{W}} = \mathbf{0} \tag{125}$$

which, when compared with Eq. (33), shows that $\mathbf{C}$, in the outlined formulations, is being evaluated according to different weighting matrices, as the rows of $\widetilde{\mathbf{W}}$ cannot be obtained as linear combinations of the rows of $\mathbf{Z}$.

7. Application of the methods to unbounded regions

Although the procedure outlined for the evaluation of rigid body displacements in the hybrid stress boundary element method is not computationally intensive, it is at least conceptually more involved than the simple use of Eq. (56), as given in the displacement-based counterpart described in Section 5. However, the developments in Subsection 6.2 contain an important contribution for the consideration of infinite domains, as none of Eqs. (76) and (112) are applicable to this topologically different type of problem. The reason is that one can exclude neither rigid body displacements nor unbalanced forces from the energy considerations when dealing with an infinite domain. Fortunately, there are some simple topological relations between the matrices obtained for a cavity in an infinite domain and the corresponding matrices for the complementary bounded domain. In fact, characterizing the matrices for an infinite domain with an upper bar (-), it is straightforward to demonstrate the following simple relations, given that the sense of integration is reversed,

$$\overline{\mathbf{G}} = -\mathbf{G}, \qquad \overline{\mathbf{R}} = -\mathbf{R}, \qquad \overline{\mathbf{L}} = -\mathbf{L} \tag{126}$$

and the less straightforward relations, which involve singular integral considerations:

$$\overline{\mathbf{H}} = \mathbf{I} - \mathbf{H}, \qquad \overline{\mathbf{F}} = \mathbf{U}^* - \mathbf{F}. \tag{127}$$

From Eq. (127), one obtains by adding $\mathbf{WC}$ to both sides:

$$\mathbf{F} + \left(\overline{\mathbf{F}} + \mathbf{WC}\right) = \left(\mathbf{U}^* + \mathbf{WC}\right). \tag{128}$$

In the hybrid *stress* boundary element method, one multiplies all terms of Eq. (128) by $\mathbf{V}$ and, observing Eqs. (112) and (121), obtains:

$$\left(\overline{\mathbf{F}} + \mathbf{WC}\right)\mathbf{V} = \mathbf{0} \tag{129}$$

which is the orthogonality condition required to evaluate the coefficients about the main diagonal of the non-singular matrix $\overline{\mathbf{F}}$.

In the hybrid *displacement* boundary element method, one multiplies all terms of Eq. (128) by $\mathbf{Y}$ and, observing Equation (76) and the counterpart of Eq. (121),

$$\left(\mathbf{U}^* + \mathbf{WC}\right)\mathbf{Y} = \mathbf{0} \tag{130}$$

which per se is not required in the formulation, and obtains:

$$\left(\overline{\mathbf{F}} + \mathbf{WC}\right)\mathbf{Y} = \mathbf{0}. \tag{131}$$

8. A comparative spectral analysis of the methods outlined

The three methods presented in this paper are schematized in Figs. 5, 6 and 7, as concerning their topological properties. One readily identifies all types of transformations performed between the different coordinate systems, as outlined in Sections 3, 5 and 6, taking into account the bases $\mathbf{V}$, $\mathbf{Y}$, $\mathbf{W}$ and $\mathbf{Z}$ of inadmissible quantities. All transformations are physically interpreted. Moreover, all primary nodal parameters of the interpolated fields are identified in brackets, according to which one can represent the final results both in the domain and along the boundary.

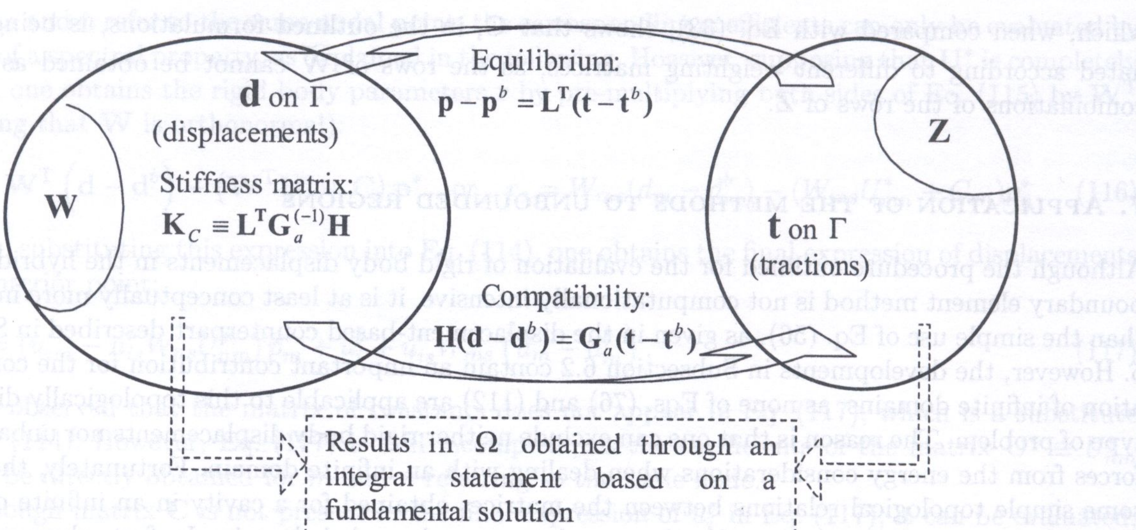

Fig. 5. Transformations carried out in the conventional boundary element method

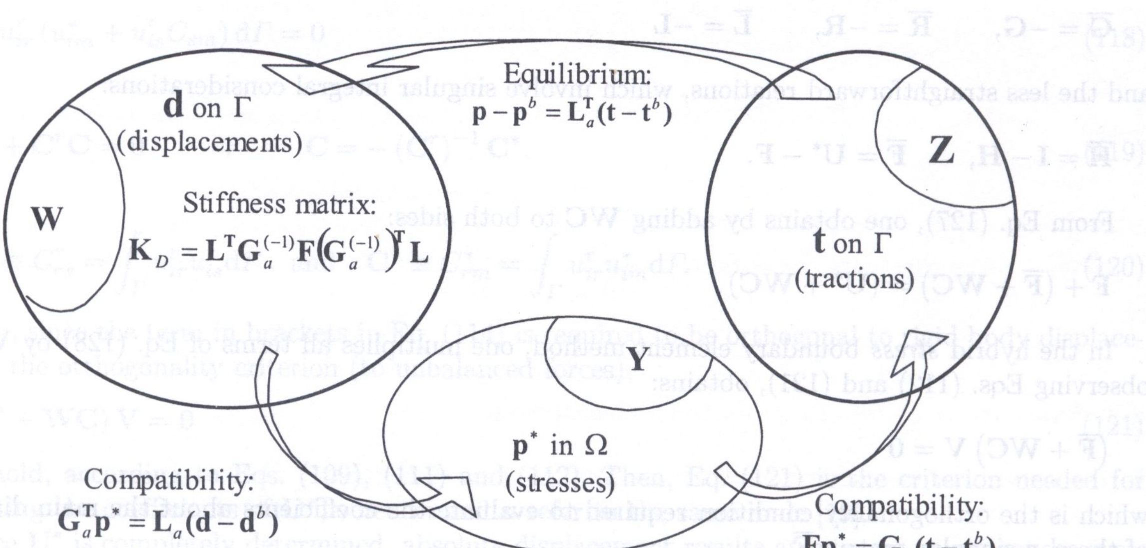

Fig. 6. Transformations carried out in the hybrid displacement boundary element method

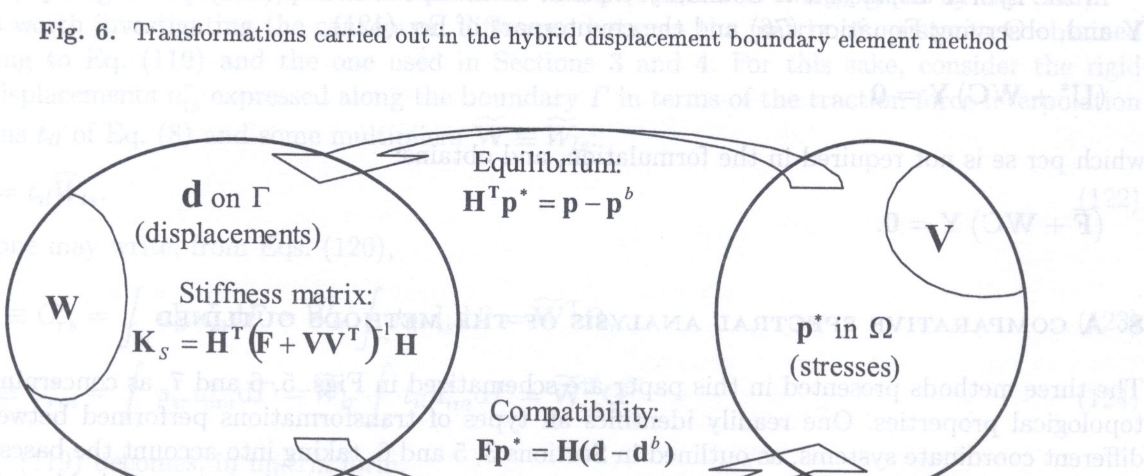

Fig. 7. Transformations carried out in the hybrid stress boundary element method

9. Conclusions

For the sake of brevity, comparative numerical results could not be considered in this article. All three formulations perform equivalently, in terms of both accuracy and spectral properties, provided that one considers the admissible matrix $\mathbf{G}_a$ and proceeds as outlined in Section 3. Use of the inconsistent matrix $\mathbf{G}$ may lead to unreliable results, in case of ill conditioning. Moreover, it is shown how to derive correctly a spectrally consistent stiffness matrix in the CBEM.

All considerations in this paper are stated primarily for a finite, simply connected domain. For an infinite domain, the modifications briefly reported in Section 7 lead to the required extensions, although some new considerations have to be added. Multiply connected domains may always be taken into account by superposing domains [19].

The author hopes to have accomplished his task: a) to demonstrate that in all boundary element formulations one has to deal with singular matrices and generalized inverses; b) to outline two variational counterparts of the collocation boundary element method and present their conceptual affinities and differences. A not unremarkable conclusion is that both the conventional and the hybrid displacement boundary element methods have required some conceptual improvements in their formulations, in order to become completely consistent. These improvements (and some corrections) concern particularly the consideration of body forces, which are much more elegant and easier to implement in the proposed formulations.

The proposed implementation for body forces is particularly advantageous for time-dependent problems [19, 20]. It is worth mentioning that a simplification of the hybrid boundary element methods is possible, in which one avoids the computationally intensive evaluation of the flexibility matrix $\mathbf{F}$, at the cost of loosing the complete variational consistency of the methods [19].

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