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The Multipole method for the Laplace equation in domains with polyhedral corners

Sergey L. Skorokhodov, Vladimir I. Vlasov
Dorodnicyn Computing Centre of Russian Acad. Sci.,
40 Vavilova Str., Moscow 119991;
skor@ccas.ru, vlasov@ccas.ru

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A new analytic–numerical method has been developed for solving the Laplace equation in domains with cones of arbitrary base, in particular with polyhedral corners. The solution is represented as an expansion involving singular functions (the Multipoles), which play the role of basic functions. The method enables to find these functions explicitly and to compute efficiently their singularity exponents. The method possesses exponential rate of convergence and provides precise computation of the solution, its derivatives and intensity factors at the edges and at the corner point. In addition, an asymptotic expansion of the solution near the edges of polyhedral corner has been obtained.

1. Introduction

Solutions of elliptic boundary value problems (BVPs) in domains with cones or polyhedral corners have singularities at vertices and edges of the cones and corners [1–6]. Because of the singularities, standard numerical techniques meet essential difficulties resulting in the loss of accuracy and convergence [7–11]. The elaboration of effective computational methods for solving these problems became a challenging issue [9–16].

In this work, we present a new effective analytic-numerical method for solving BVPs for the Laplace equation in domains with cones of arbitrary base (in particular, polyhedral corners); its idea was communicated at the conference [17], and first numerical results at the conference [18]. This method represents a generalization of the Multipole method, previously developed in [19–23] for solving a certain class of 2D and 3D elliptic BVPs in domains of complex shape.

The principle underlying our method consists in using a system of basic functions Ψ_k that conform to the structure of the solution near the conical surfaces of the boundary. Functions Ψ_k are sometimes called singular functions [14], [2–26]. We call them Multipoles due to their similarity to ordinary Multipoles, known in the theory of potential [27]. Such systems possess good approximating properties. Most important is the fact that these basic functions are expressed in explicit analytic form in terms of special functions.

By virtue of these features our method proves most effective for precise computation of the solution and its derivatives up to the conical surfaces of the boundary, in spite of the singularities mentioned above. An important advantage of our method is that it yields values of intensity factors at the vertex and edges of polyhedral corner along with the solution itself.

The solution of the BVP is represented as an expansion in terms of the above basic system. Unlike the finite element method, in our method the basic functions are defined on the domain as a whole, so they are global rather than local. That is why our method does not need any mesh.

Due to the analytic representation of the solution, our method proves favourable for qualitative analysis of the solution and its characteristics; in particular, we have found an asymptotic expansion of the solution near the edges of polyhedral corner.

2. Statement of the problem. Solvability and regularity

2.1. Domains K and Ω

Let (x_1, x_2, x_3) be Cartesian and (r, θ, φ) spherical coordinates of a point x in space $\mathbf{R}^3$. Coordinates of those systems are related by well-known formulas.

Denote by $\mathbf{S}^2 \stackrel{\text{def}}{=} \{(r, \theta, \varphi) : r = 1, \theta \in [0, \pi], \varphi \in [0, 2\pi)\}$ the unit sphere and by $\mathbf{B}^3$ the unit ball in $\mathbf{R}^3$ (evidently $\partial\mathbf{B}^3 = \mathbf{S}^2$). Points $\theta = 0$ and $\theta = \pi$ on $\mathbf{S}^2$ are called the North Pole $\mathcal{P}_N$ and the South Pole $\mathcal{P}_S$, respectively.

Consider a Lipschitz piecewise smooth contour $\mathcal{L}$ on the sphere $\mathbf{S}^2$ with no self-crossing points, dividing $\mathbf{S}^2$ into two domains, one of which contains $\mathcal{P}_S$ and another $\mathcal{P}_N$. The domain containing $\mathcal{P}_N$ and bounded by $\mathcal{L}$ is denoted by $\mathcal{S}$.

The domain $\mathbf{K} \subset \mathbf{R}^3$ defined by the formula $\mathbf{K} \stackrel{\text{def}}{=} \{(r, \theta, \varphi) : r \in (0, \infty), (\theta, \varphi) \in \mathcal{S}\}$ is an (infinite) cone with base $\mathcal{S}$, its boundary being the conical surface $\partial\mathbf{K} = \{(r, \theta, \varphi) : r \in (0, \infty), (\theta, \varphi) \in \mathcal{L}\}$.

Consider an important instance of the cone $\mathbf{K}$ when it presents a trihedral corner with its three faces being plane angles with common vertex $\{0\}$ and with values of the angles being equal to $\pi\alpha$, where $\alpha \in (0, 2/3]$. Denote by $\mathbf{K}^\alpha$ this trihedral corner, by $\mathcal{S}^\alpha$ its base, and by $\mathcal{L}^\alpha$ the contour of this base. In this instance, the equation of contour $\mathcal{L}^\alpha$ is written in the form

$$\mathcal{L}^\alpha = \{(\theta, \varphi) : \theta = \theta(\varphi), \varphi \in [0, 2\pi)\}, \qquad \theta(\varphi) = \begin{cases} T(\varphi + 2\pi/3); & \varphi \in [-\pi, -\pi/3], \\ T(\varphi); & \varphi \in [-\pi/3, \pi/3], \\ T(\varphi - 2\pi/3); & \varphi \in [\pi/3, \pi], \end{cases} \tag{1}$$

with function $T(\varphi)$ defined by the formula

$$T(\varphi) = \arccos\left[-\frac{\cos\varphi}{\sqrt{q + \cos^2\varphi}}\right], \qquad q = \frac{1 - \cos\pi\alpha}{2(1 + 2\cos\pi\alpha)}. \tag{2}$$

Then base $\mathcal{S}^\alpha$ is given as follows

$$\mathcal{S}^\alpha = \{(\theta, \varphi) : \theta \in [0, \theta(\varphi)), \varphi \in [0, 2\pi)\}.$$

Remark 4. *For all $\alpha \in (0, 2/3)$, the function $\theta(\varphi)$ presenting in the Eq. (1) of contour $\mathcal{L}^\alpha$ satisfies the inequality $\forall\varphi \;\; \theta(\varphi) > \pi/2$, so that cone $\mathbf{K}^\alpha$ always contains the half-space $\mathbf{R}^3_+ \stackrel{\text{def}}{=} \{(r, \theta, \varphi) : r \in (0, \infty), \theta \in [0, \pi/2), \varphi \in [0, 2\pi)\}$. If $\alpha = 2/3$, then $\theta(\varphi) \equiv \pi/2$, and $\mathbf{K}^{2/3} = \mathbf{R}^3_+$.*

It worth to be mentioned that values $\pi\beta$ of dihedral angles between faces of cone $\mathbf{K}^\alpha$ are related to the quantity $\pi\alpha$ by the formula

$$\cos\pi\beta = \cos\pi\alpha \,/\, (1 + \cos\pi\alpha), \qquad \alpha \in (0, 2/3]. \tag{3}$$

In accordance with Remark 1, if $\alpha \in (0, 2/3)$, then $\beta > 1$, and if $\alpha = 2/3$, then $\beta = 1$.

The BVP is being solved in a domain $\Omega \subset \mathbf{K}$ homeomorphic to $\mathbf{B}^3$ with Lipschitz piecewise smooth boundary $\partial\Omega$. By definition, boundary $\partial\Omega$ consists of the two disjoint parts: γ and Γ, where γ is a closure of a simply-connected domain of the conical surface with the vertex of $\partial\mathbf{K}$ being an interior point of γ, and $\Gamma \subset \mathbf{K}$ is a simply-connected domain on a certain piecewise smooth surface. Note that $\mathbf{K}$ is an extension of Ω through Γ.

Let the surface Γ be divided by a Lipschitz piecewise smooth curve or contour into two domains: $\mathcal{D}$ and $\mathcal{N}$; the boundary condition of the Dirichlet type is to be set on $\mathcal{D}$ and the Neumann type on $\mathcal{N}$.

2.2. The formulation of the BVP with mixed Dirichlet–Neumann boundary conditions

Consider the following BVP for the Laplace equation in the domain Ω :

$$\Delta\psi = 0 \qquad \text{in } \Omega, \tag{4}$$

with mixed Dirichlet–Neumann type conditions

$$\psi\Big|_{\gamma} = 0, \qquad \psi\Big|_{\mathcal{D}} = h_{\mathcal{D}}, \qquad \frac{\partial\psi}{\partial\nu}\Big|_{\mathcal{N}} = h_{\mathcal{N}} \tag{5}$$

on the boundary $\partial\Omega = \gamma \cup \Gamma$, where $\partial/\partial\nu$ is a normal derivative. We shall use the notation $h(x)$ defined by equalities

$$h(x) = h_{\mathcal{D}}(x), \qquad x \in \mathcal{D}; \qquad h(x) = h_{\mathcal{N}}(x), \qquad x \in \mathcal{N}. \tag{6}$$

In order to formulate a generalized statement of the BVP (4), (5), appropriate Sobolev spaces are introduced. Denote by $\overset{\circ}{W}{}_2^1(\Omega, \gamma)$ a subspace of $W_2^1(\Omega)$ consisting of functions having zero trace on γ. Similarly, define the space $\overset{\circ}{W}{}_2^1(\Omega, \gamma \cup \mathcal{D})$ as a subspace of $W_2^1(\Omega)$ consisting of functions with zero trace on $\gamma \cup \mathcal{D}$.

Let A be a subdomain of the boundary $\partial\Omega$, and let a be a subdomain of A. Denote by $\overset{\circ}{W}{}_2^{1/2}(A, a)$ a subspace of the Sobolev–Slobodetskii space $W_2^{1/2}(A)$ consisting of functions vanishing a.e. on a. Only the particular cases of the latter spaces $\overset{\circ}{W}{}_2^{1/2}(\gamma \cup \mathcal{D}, \gamma)$ and $\overset{\circ}{W}{}_2^{1/2}(\partial\Omega, \gamma \cup \mathcal{D})$ are to be employed below. The so called negative space $\overset{\circ}{W}{}_2^{-1/2} = (\partial\Omega, \gamma \cup \mathcal{D})$ is defined as a conjugate space to $\overset{\circ}{W}{}_2^{1/2}(\partial\Omega, \gamma \cup \mathcal{D})$.

The boundary data $h_{\mathcal{D}}$ and $h_{\mathcal{N}}$ in conditions (5) are required to belong to the spaces

$$h_{\mathcal{D}} \in \overset{\circ}{W}{}_2^{1/2}(\gamma \cup \mathcal{D}, \gamma), \qquad h_{\mathcal{N}} \in \overset{\circ}{W}{}_2^{-1/2}(\partial\Omega, \gamma \cup \mathcal{D}). \tag{7}$$

A generalized solution of BVP (4), (5) is understood to be a function $\psi \in \overset{\circ}{W}{}_2^1(\Omega, \gamma)$ satisfying boundary condition $\psi|_{\mathcal{D}} = h_{\mathcal{D}}$ and the integral identity

$$\int_{\Omega} (\nabla\psi, \nabla\eta)\, \mathrm{d}x = \int_{\mathcal{N}} h_{\mathcal{N}}\, \eta\, \mathrm{d}s$$

for all test-functions $\eta \in \overset{\circ}{W}{}_2^1(\Omega, \gamma \cup \mathcal{D})$, where the notation $(\,.\,,\,.\,)$ stands for the inner product in Euclidean space $\mathbf{R}^3$.

Solvability of the formulated BVP is guaranteed by the following

Theorem 1. *For any $h_{\mathcal{D}}$ and $h_{\mathcal{N}}$ satisfying (7) there exists a unique generalized solution $\psi \in \overset{\circ}{W}{}_2^1(\Omega, \gamma)$ of the problem (4), (5).*

It is clear that Theorem 1 admits a standard proof which reduces to the Riesz representation theorem and follows well-known patterns (see e.g. [28]). Outside the boundary's singularities, regularity of the generalized solution of (4), (5) is covered by the standard theory of elliptic BVPs. Namely, the generalized solution is infinitely differentiable at any interior point $x \in \Omega$. At any interior point of γ the generalized solution is differentiable as many times as the smoothness of γ at this point allows. Omitting the details, we just mention that regularity of the generalized solution at boundary points $x \in \mathcal{D}$ and $x \in \mathcal{N}$ depends on the smoothness of boundary surface Γ and boundary data $h_{\mathcal{D}}$, $h_{\mathcal{N}}$.

3. Construction of the system of basic functions (the Multipoles)

3.1. Reduction to a spectral problem for the Beltrami–Laplace operator in the domain $\mathcal{S}$ on the sphere.

Our goal consists in constructing a system of functions Ψ_k (the Multipoles) that possess good approximation properties and conform to the structure of the solution near the conical surfaces, which contain singularities (the vertex and edges). The basic functions are defined on the whole cone domain $\mathbf{K}$. The desired properties of these functions require the following conditions to be met:

- functions Ψ_k identically satisfy the Laplace equation in $\mathbf{K}$;
- they identically meet the homogeneous Dirichlet condition $\Psi_k = 0$ on $\partial\mathbf{K}$;
- they constitute an orthogonal basis in $L_2(\mathcal{S})$.

The Multipoles are represented in the form

$$\Psi_k(r,\,\theta,\,\varphi) = r^{\mu}\, U(\mu;\theta,\varphi), \qquad \mu = \mu(k)\,; \qquad k = 1,\,2,\ldots \tag{8}$$

Thus $U(\mu(k);\,\theta,\,\varphi) = U_k$ are eigenfunctions with eigenvalues $\mu(k)$ for the Laplace–Beltrami operator in the domain $\mathcal{S}$ on the unit sphere

$$\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2 U}{\partial\varphi^2} + \mu\,(\mu+1)\,U = 0 \qquad \text{in } \mathcal{S}\,, \tag{9}$$

with homogeneous Dirichlet condition on $\mathcal{L}$:

$$U\Big|_{\mathcal{L}} = 0\,. \tag{10}$$

Denote by $\overset{\circ}{W}{}_2^1(\mathcal{S})$ a subspace of $W_2^1(\mathcal{S})$ consisting of functions having zero trace on $\mathcal{L}$. A generalized solution of BVP (9), (10) is understood to be a function $U\in \overset{\circ}{W}{}_2^1(\mathcal{S})$ satisfying the integral identity

$$\int_{\mathcal{S}} (\,\nabla_{\mathcal{S}}\, U,\, \nabla_{\mathcal{S}}\, V\,)\, ds = \mu\,(\mu+1)\int_{\mathcal{S}} U\,V\,ds \qquad \forall\, V \in \overset{\circ}{W}{}_2^1(\mathcal{S}), \tag{11}$$

where $\nabla_{\mathcal{S}}$ stands for a tangential component to $\mathcal{S}$ of the gradient ∇. Note that an inner product

$$[U,\,V]_{\mathcal{S}} \overset{\text{def}}{=} \int_{\mathcal{S}} (\,\nabla_{\mathcal{S}} U,\, \nabla_{\mathcal{S}} V\,)\, ds$$

induces an equivalent norm on $\overset{\circ}{W}{}_2^1(\mathcal{S})$.

Theorem 2. *For a spectral problem (9), (10) there exists a denumerable set of generalized solutions* $U = U_k \in \overset{\circ}{W}{}_2^1(\mathcal{S})$, $\mu = \mu(k)$, $k = 1,\,2,\ldots$ *The eigenvalues* $\mu(k)$ *have no finite limit points, and* $\mu(k) \to \infty$ *as* $k \to \infty$. *To each eigenvalue there corresponds at most a finite number of generalized eigenfunctions* $U_k \in \overset{\circ}{W}{}_2^1(\mathcal{S})$. *The eigenfunctions* $\{U_k\}$ *form a basis in* $L_2(\mathcal{S})$ *and* $W_2^1(\mathcal{S})$, *which is orthonormal in* $L_2(\mathcal{S})$ *and orthogonal with respect to the inner product* $[\,\cdot\,,\,\cdot\,]_{\mathcal{S}}$.

It is clear that Theorem 2 admits a standard proof which follows the pattern of [29].

Remark 5. *In accordance with Theorem 2, all eigenvalues* $\mu(k)$, $k = 1,\,2,\,\ldots$ *can be enumerated in order of their nondecreasing; each multiple eigenvalue should be counted according to its multiplicity. Such renumbering establishes a one-to-one correspondence between eigenvalues* $\mu(k)$ *and eigenfunctions* U_k .

3.2. Solution of the spectral problem

In what follows, we restrict ourselves to the case of contours $\mathfrak{L}$ being star-shaped on $\mathbf{S}^2$ when $\mathfrak{L}$ can be represented in the form

$$\mathfrak{L} = \{(\theta, \varphi) : \theta = \theta(\varphi), \quad \theta(\varphi) \in C(-\infty, +\infty), \quad \theta(\varphi) = \theta(\varphi + 2\pi)\}.$$

The eigenfunctions of the problem (9), (10) are constructed using the system of complex–valued functions: $\{u^m(\mu; \theta, \varphi)\}_{m=0}^{\infty}$ defined by the formula:

$$u^m(\mu; \theta, \varphi) \stackrel{\text{def}}{=} P_\mu^m(\cos\theta)\, e^{im\varphi}, \tag{12}$$

where $\mathrm{P}_\mu^m(t)$ are associated Legendre functions on the cut [30]. For short, in complicated expressions we reduce the relations (12) to $u^m(\mu)$.

Note that if $\mathbf{K}$ is a circular cone, i.e. $\mathfrak{L}$ is a circumference $\{\theta = \theta_0 = \text{const}\}$, then

$$u_n^{m+} \stackrel{\text{def}}{=} \operatorname{Re} u^m(\mu; \theta, \varphi) \quad \text{and} \quad u_n^{m-} \stackrel{\text{def}}{=} \operatorname{Im} u^m(\mu; \theta, \varphi)$$

are eigenfunctions of the problem (9), (10) with eigenvalue $\mu = \mu_n^m$ being the root of number $n\,(n = 1, 2, \dots)$ of the equation $\mathrm{P}_\mu^m(\cos\theta_0) = 0$. Taking this fact into account, we rename and renumber eigenvalues $\mu(k)$ as μ_n^m and eigenfunctions $U_k = U(\mu(k); \theta, \varphi)$ as $U_n^{m+}(\theta, \varphi)$ and $U_n^{m-}(\theta, \varphi)$.

Let us represent the desired eigenfunctions in the form of expansions in terms of functions (12):

$$U_n^{m\pm} = \operatorname{Re}\sum_{l=0}^{\infty} A_n^{m,l\pm}\, u^{m+l}(\mu), \qquad A_n^{m,0+} = 1, \qquad A_n^{m,0-} = i. \tag{13}$$

Observe that functions (13) with any coefficients identically satisfy the Eq. (9). Unknown eigenvalues μ_n^m and coefficients $A_n^{m,l\pm}$ in representation (13) should be found from the boundary condition (10) on the contour $\mathfrak{L}$.

We shall make it in the following way. Functions $U_n^{m\pm}(\theta, \varphi)$ are sought as a limit

$$U_n^{m\pm}(\theta, \varphi) = \lim_{M\to\infty} U_n^{m\pm}(M; \theta, \varphi)$$

of consequent approximations $U_n^{m\pm}(M; \theta, \varphi)$ written in the form of finite sums (13), with coefficients depending on the length M of approximation, i.e.

$$U_n^{m\pm}(M; \theta, \varphi) = \operatorname{Re}\sum_{l=0}^{M} A_n^{m,l\pm}(M)\, u^{m+l}, \qquad A_n^{m,0+}(M) = 1, \qquad A_n^{m,0-}(M) = i. \tag{14}$$

Coefficients $A_n^{m,l\pm}(M)$ and approximate eigenvalues $\mu_n^{m\pm}(M)$ are determined by substituting $U_n^{m\pm}(M)$ into the boundary condition (10) and by projecting the result onto the system of trigonometric functions $\exp(iq\varphi)$:

$$\Big(U_{n,ex}^{m\pm}(M),\ \exp(iq\varphi)\Big)_{\mathfrak{L}} = 0, \tag{15}$$

where $q = m, \dots m + M$, and $(f_1, f_2)_{\mathfrak{L}}$ is the inner product in $L_2(\mathfrak{L})$. Substituting representation (14) into relation (15) we obtain a system of linear equations with respect to coefficients $A_n^{m,l\pm}(M)$:

$$\mathcal{D}^m(\mu)\,\mathcal{Z} = 0, \tag{16}$$

where

$$\mathcal{Z} = \left(A_n^{m,0\pm}(M),\, A_n^{m,1\pm}(M), \ldots, A_n^{m,M\pm}(M)\right)^T$$

is a vector of the coefficients. Elements of matrix $\mathcal{D}^m(\mu)$ of system (16) are expressed as integrals over contour $\mathcal{L}$ of products of functions (12) and trigonometric functions $\exp(i\,q\,\varphi)$.

In order to find a nontrivial solutions of homogeneous system (16) we equate the determinant of its matrix to zero

$$\det \mathcal{D}^m(\mu) = 0. \tag{17}$$

Then eigenvalue $\mu_n^m(M)$ is a root of number n ($n = 1, 2, \ldots$) of Eq. (17).

The performed numerical experiments showed that the approximate eigenvalues and eigenfunctions converge to the exact ones. Namely, there hold the relations:

1) for any compact $E \subset \mathcal{S}$ it holds

$$U_n^{m\pm}(M; \theta, \varphi) \Longrightarrow U_n^{m\pm}(\theta, \varphi) \quad \text{for} \quad \{\theta, \varphi\} \in E;$$

2) for all coefficients in (14) and all eigenvalues it holds

$$A_n^{m,l\pm}(M) \longrightarrow A_n^{m,l\pm} \quad \text{and} \quad \mu_n^m(M) \longrightarrow \mu_n^m \quad \text{as} \quad M \to \infty.$$

3.3. Computation of integrals of frequently oscillating functions

One of important computational problems arising in the described algorithm is the calculation of elements of matrix $\mathcal{D}^m(\mu)$ of system (16); those elements are expressed in the form of integrals of the following type:

$$\int_{\mathcal{L}} P_\mu^a\big(\cos\theta(\varphi)\big)\exp(ib\varphi)\,d\varphi\,, \tag{18}$$

where $\theta(\varphi)$ is an equation of the contour; a and b are nonnegative integers, possibly very large. So, (18) are integrals with frequently oscillating integrand; effective computation of those integrals is a well-known challenging problem. A special analytic-numerical method has been developed for computation of such integrals. This method represents integrals (18) as exponentially convergent series involving integrals $\int_0^{\pi/2} \cos^\alpha t \cos\beta t\, dt$ and related ones, which we have computed explicitly. For instance,

$$\int_0^{\pi/2} \cos^\alpha t \cos\beta t\, dt = \pi\,(1+\alpha)\,2^{-1-\alpha}\left[B\Big(1+(\alpha+\beta)/2,\; 1+(\alpha-\beta)/2\Big)\right]^{-1},$$

where $B(x, y)$ is Beta–function [30].

3.4. The Multipoles Ψ_k

In accordance with Theorem 2 and Remark 2, all eigenvalues $\mu_n^m(M)$ can be enumerated as $\mu(k)$, $k = 1, 2, \ldots$, in order of their nondecreasing; each multiple eigenvalue should be counted according to its multiplicity. Thus, there arises respective numeration of the eigenfunctions $U_n^{m\pm}(\theta, \varphi)$ as $U(\mu(k); \theta, \varphi) = U_k(\theta, \varphi)$ and, as a consequence, respective numeration of the Multipoles

$$\Psi_k(r, \theta, \varphi) = r^{\mu(k)}\, U_k(\theta, \varphi); \tag{19}$$

this manner of their numeration had already appeared in (8). Remind that in line with Remark 2, each eigenfunction U_k (and, consequently, each Multipole Ψ_k) corresponds to one and only one eigenvalue $\mu(k)$.

Along with (19) it will be useful other representation for the Multipoles $\Psi_k = \Psi_n^{m\pm}$ in the following form:

$$\Psi_n^{m\pm}(r, \theta, \varphi) = r^{\mu_n^m} U_n^{m\pm}(\theta, \varphi). \tag{20}$$

3.5. Numerical results

The method for solving the spectral problem (9), (10) described in Sec. 3.2 has been numerically realized for the case, when the domain $\mathcal{S}$ on the sphere $\mathbf{S}^2$ presents $\mathcal{S}^\alpha$ with its boundary contour $\mathcal{L}^\alpha$ defined by the relations (1), (2). In example in numerical realization we considered the case of the trihedral corner $\mathbf{K}^\alpha$ whose faces' plane angles at the vertex $\{0\}$ are equal to $\pi\alpha$, and dihedral angles between faces of $\mathbf{K}^\alpha$ are equal to $\pi\beta$ related to $\pi\alpha$ by (3). Particular attention must be given to the case of cone $\mathbf{K}^{1/2}$ being the space $\mathbf{R}^3$ with cutted out octant because of its importance, note that $\beta = 3/2$ for this cone. Cone $\mathbf{K}^{1/2}$ was considered in many works (see, for example, [2, 4, 9–11] and references herein).

In implementation of the method described in Sec. 3.2 we used the Gauss elimination method for solving the system (16). Numerous computational experiments have been performed. The experiments showed that the method for solving the problem (9), (10) has the first order rate of convergence in the following sense. Denote variations of eigenvalues and coefficients $A_n^{m,\,l\,\pm}(M)$ on Mth step of the algorithm as follows

$$\Delta\,\mu_n^m(M) = \left|\mu_n^m(M+1) - \mu_n^m(M)\right|, \qquad \Delta\,A_n^{m,l\pm}(M) = \left|A_n^{m,l\pm}(M+1) - A_n^{m,l\pm}(M)\right|.$$

From our experiments we obtained

$$\mu_n^m(M) \sim \mu_n^m + a\,M^{-1} \quad \text{and} \quad A_n^{m,l\pm}(M) \sim A_n^{m,l\pm} + a'\,M^{-1} \quad \text{as} \quad M \to \infty \tag{21}$$

with some factors a, a'. This result is well illustrated by Fig. 1, where the graph of the dependence $\ln\Delta\,\mu_1^0(M)$ versus $\ln M$ is given. Straight-line shape of this graph just corresponds to the asymptotic (21). The numerical data for this graph were obtained for $\alpha = 1/2$; numerical experiments for other values of α have given the same result.

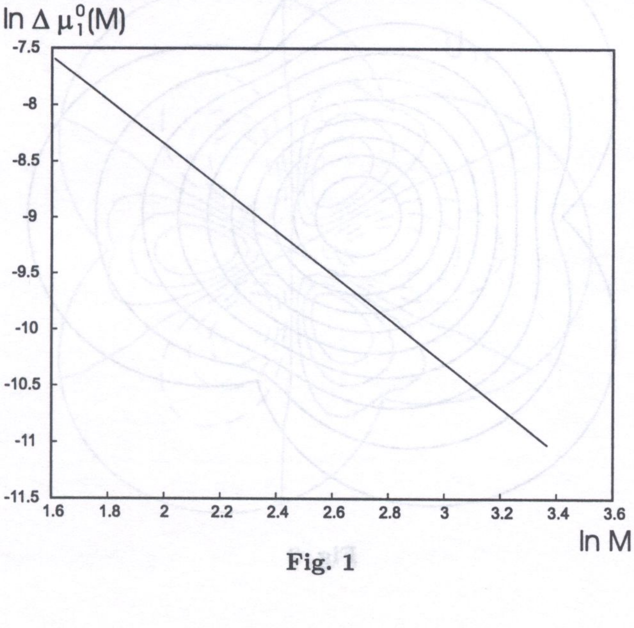

Fig. 1

Let present now the first 14 eigenvalues for $\pi\alpha = \pi/2$.

$$\mu(1) = \mu_1^0 = 0.454167; \qquad \mu(14) = \mu(15) = \mu_2^2 = 3.33241;$$

$$\mu(2) = \mu(3) = \mu_1^1 = 1.23021; \qquad \mu(16) = \mu(17) = \mu_3^1 = 3.79278;$$

$$\mu(4) = \mu(5) = \mu_2^0 = 1.78349; \qquad \mu(18) = \mu(19) = \mu_1^4 = 4.03836;$$

$$\mu(6) = \mu(7) = \mu_1^2 = 2.12039; \qquad \mu(20) = \mu(21) = \mu_2^3 = 4.24952;$$

$$\mu(8) = \mu(9) = \mu_2^1 = 2.51627; \qquad \mu(22) = \mu(23) = \mu_4^0 = 4.47577;$$

$$\mu(10) = \mu(11) = \mu_1^3 = 3.07781; \qquad \mu(24) = \mu(25) = \mu_3^2 = 4.57123;$$

$$\mu(12) = \mu(13) = \mu_3^0 = 3.16648; \qquad \mu(26) = \mu(27) = \mu_1^5 = 5.02247.$$

We must note that for the case under consideration ($\alpha = 1/2$) the first eigenvalue $\mu(1)$ has been computed in the work [9] with an accuracy of four decimal places: $\mu(1) = 0.4542$.

And now let us turn to eigenfunctions corresponding to some of the above eigenvalues. A representation for these functions is given in the form of distribution of their level lines in stereographic projection; we mean the projection of the sphere $\mathbf{S}^2$ from its South pole $\mathcal{P}_S$ onto the plane (x_1, x_2). Note that co-ordinates (x_1, x_2) introduced in the beginning of Sec. 2.1 are related to spherical angles (θ, φ) by the well-known formulas:

$$x_1 = \operatorname{ctg}(\theta/2)\cos\varphi, \qquad x_2 = \operatorname{ctg}(\theta/2)\sin\varphi.$$

Some eigenfunctions for the case $\alpha = 1/2$ are shown in the above representation in Fig. 2–6. Notice that our eigenfunctions U_k are even with respect to co-ordinate x_2 for odd k, and conversely, eigenfunctions U_k are odd for even k. The function $U_1 = U_1^{0+}$ with $\mu(1) = \mu_1^0 = 0.454167$ is shown in Fig. 2. The function $U_6 = U_1^{2-}$ with $\mu(6) = \mu_1^2 = 2.12039$ is shown in Fig. 3. The function $U_7 = U_1^{2+}$ with $\mu(7) = \mu_1^2 = 2.12039$ is shown in Fig. 4. The function $U_{11} = U_1^{3+}$ with $\mu(11) = \mu_1^3 = 3.07781$ is shown in Fig. 5. The function $U_{13} = U_3^{0+}$ with $\mu(13) = \mu_3^0 = 3.16648$ is shown in Fig. 6.

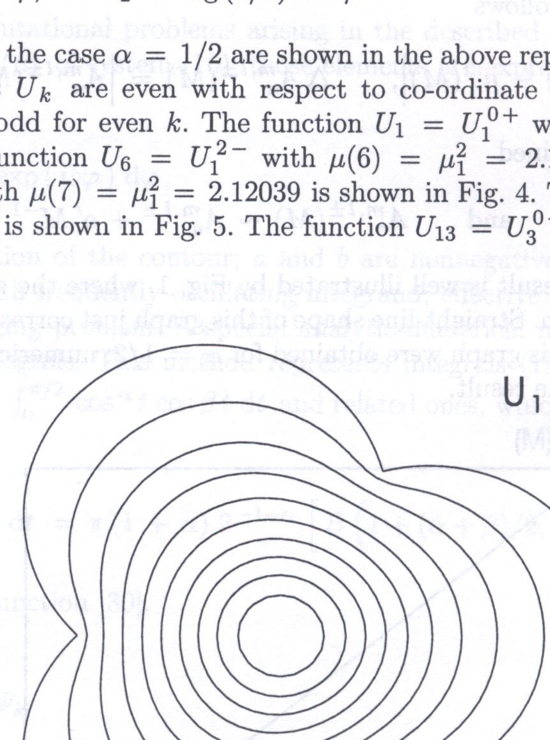

Fig. 2

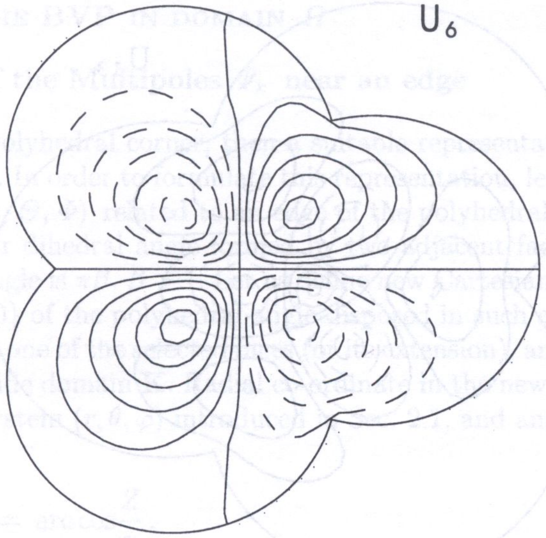

Fig. 3

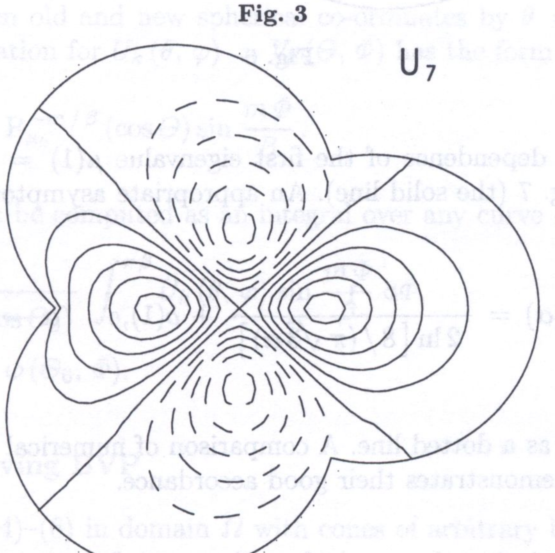

Fig. 4

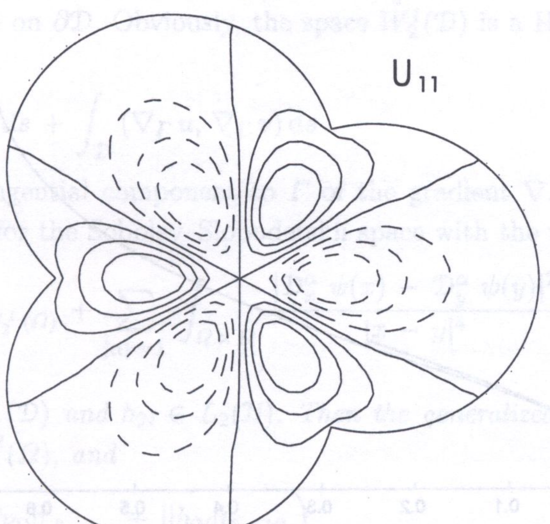

Fig. 5

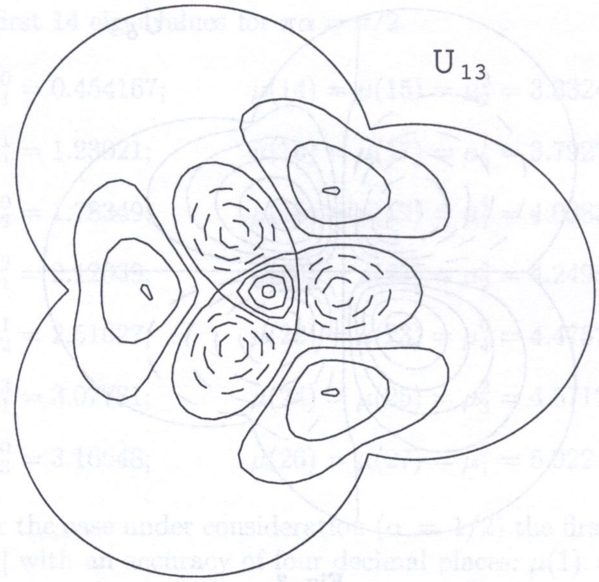

Fig. 6

For the domain $\mathbb{S}^\alpha$ the dependence of the first eigenvalue $\mu(1) = \mu_1^0$ versus $\alpha \in [0, 2/3]$ is presented as a graph in Fig. 7 (the solid line). An appropriate asymptotic relation for small α has the form

$$\mu_1^0(\alpha) = \frac{1}{2 \ln\left[8 / (\pi \sqrt{3}\alpha)\right]} + o(1), \qquad \alpha \to 0.$$

Its graph is given in Fig. 7 as a dotted line. A comparison of numerical results (solid line) and the asymptotics (dotted line) demonstrates their good accordance.

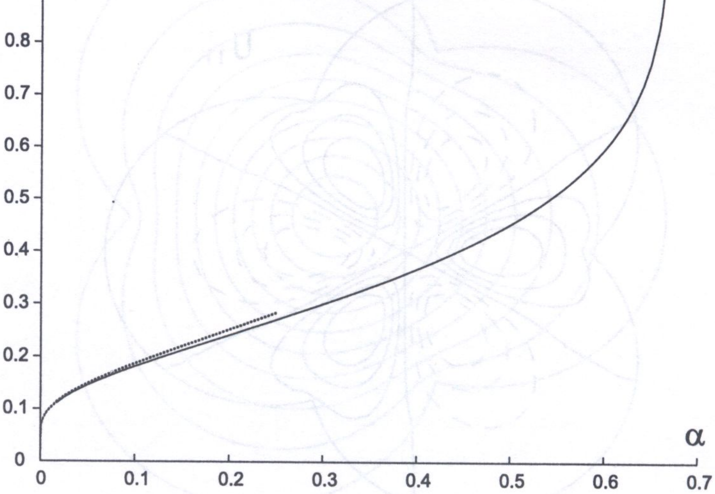

Fig. 7

4. The solution of the BVP in domain Ω

4.1. A representation of the Multipoles Ψ_k near an edge

If our cone $\mathbf{K}$ is in fact a polyhedral corner, then a suitable representation for the Multipoles (19) near the edges can be given. In order to formulate this representation, let us introduce a new system of spherical co-ordinates (r, Θ, Φ) related to an edge of the polyhedral angle.

Let us select a particular dihedral angle formed by two adjacent faces and corresponding edge. The value of the dihedral angle is $\pi\beta$, $\beta > 1$. Let us define new Cartesian co-ordinates X, Y, Z with their origin at the vertex $\{0\}$ of the polyhedral angle disposed in such a way that the selected edge lies on axis Z, axis X lies on one of the selected faces (or its extension), and axis Y is perpendicular to this face and is directed inside domain $\mathbf{K}$. Radial co-ordinate in the new system (r, Θ, Φ) coincides with the same one of the system (r, θ, φ) introduced in Sec. 2.1, and angle co-ordinates are defined by the standard formulas

$$\Phi = \arctan\frac{Y}{X}, \quad \Theta = \arccos\frac{Z}{r}.$$

Denote the relation between old and new spherical co-ordinates by $\theta = \theta(\Theta, \Phi)$, $\varphi = \varphi(\Theta, \Phi)$. Then the desired representation for $U_k(\theta, \varphi) = V_k(\Theta, \Phi)$ has the form

$$V_k(\Theta, \Phi) = \sum_{m=1}^{\infty} D_k^m \, \mathrm{P}_{\mu_k}^{-m/\beta}(\cos\Theta) \sin\frac{m\Phi}{\beta}. \tag{22}$$

Coefficients D_k^m in (22) can be computed as an integral over any curve $\{\Theta = \Theta_0 = \text{const}\} \subset \mathbb{S}$:

$$D_k^m = \frac{2}{\pi\beta\,\mathrm{P}_{\mu_k}^{-m/\beta}(\cos\Theta_0)} \int_0^{\pi\beta} U_k(\theta, \varphi) \sin\frac{m\Phi}{\beta}\, \mathrm{d}\Phi,$$

where $\theta = \theta(\Theta_0, \Phi)$, $\varphi = \varphi(\Theta_0, \Phi)$.

4.2. The method for solving BVP

Now we turn to the BVP (4)–(6) in domain Ω with cones of arbitrary base as described in Sec. 2. Note that $\partial\Omega$ may have at most a finite number of edges and conical points. Since the boundary $\partial\Omega$ is Lipschitz, a Sobolev space $\overset{\circ}{W}{}_2^1(\mathcal{D})$ is defined habitually as a subspace of $W_2^1(\mathcal{D})$ consisting of functions having zero trace on $\partial\mathcal{D}$. Obviously, the space $\overset{\circ}{W}{}_2^1(\mathcal{D})$ is a Hilbert space with the inner product

$$[u, v]_{\overset{\circ}{W}{}_2^1(\mathcal{D})} = \int_{\mathcal{D}} uv\,\mathrm{d}s + \int_{\mathcal{D}} (\nabla_\Gamma u, \nabla_\Gamma v)\,\mathrm{d}s,$$

where ∇_Γ stands for a tangential component to Γ of the gradient ∇. In the following theorem, notation $W_2^{3/2}(\Omega)$ stands for the Sobolev–Slobodetskii space with the norm

$$\|\psi\|^2_{W_2^{3/2}(\Omega)} = \|\psi\|^2_{W_2^1(\Omega)} + \sum_{|\alpha|=1} \int_{\Omega\times\Omega} \frac{|\mathcal{D}_x^\alpha \psi(x) - \mathcal{D}_y^\alpha \psi(y)|^2}{|x-y|^4}\,\mathrm{d}x\,\mathrm{d}y.$$

Theorem 3. *Let $h_{\mathcal{D}} \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ and $h_{\mathcal{N}} \in L_2(\mathcal{N})$. Then the generalized solution $\psi \in \overset{\circ}{W}{}_2^1(\Omega, \gamma)$ in Theorem 1 belongs to $W_2^{3/2}(\Omega)$, and*

$$\|\psi\|_{W_2^{3/2}(\Omega)} \leq C\Big(\|h_{\mathcal{D}}\|_{\overset{\circ}{W}{}_2^1(\mathcal{D})} + \|h_{\mathcal{N}}\|_{L_2(\mathcal{N})}\Big)$$

with constant $C > 0$ depending only on Ω.

Due to the embedding $W_2^{3/2}(\Omega)$ into $W_2^1(\partial\Omega)$, the trace on $\partial\Omega$ of the generalized solution $\psi \in W_2^{3/2}(\Omega)$ in Theorem 3 belongs to $W_2^1(\partial\Omega)$. Denote by $H(\Gamma)$ a space of all generalized solutions $\psi \in \overset{\circ}{W}{}_2^1(\Omega, \gamma) \cap W_2^{3/2}(\Omega)$ in Theorem 3 with boundary data $h_{\mathcal{D}} \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ and $h_{\mathcal{N}} \in L_2(\mathcal{N})$. Clearly, Theorem 3 implies that $H(\Gamma)$ is a Hilbert space with the inner product

$$[u, v]_H = \int_{\mathcal{D}} u\, v\, \mathrm{d}s + \int_{\mathcal{D}} (\nabla_\Gamma u, \nabla_\Gamma v)\, \mathrm{d}s + \int_{\mathcal{N}} \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu}\, \mathrm{d}s.$$

For $u \in W_2^{3/2}(\Omega)$, existence of the trace $\nabla_\Gamma u \in L_2(\mathcal{D})$ is guaranteed by the embedding of $W_2^{3/2}(\Omega)$ into $W_2^1(\partial\Omega)$. Notice that existence of the trace $\partial u/\partial\nu \in L_2(\mathcal{N})$ is guaranteed only for the functions $u \in H(\Gamma)$ by virtue of Theorem 3.

For basic functions $\{\Psi_k\}$ constructed in Sec. 3 holds

Theorem 4. *The traces on Γ of the basic functions $\{\Psi_k\}$ form a complete system in $H(\Gamma)$ which is minimal.*

Proof of the completeness in Theorem 4 is based on the approximation theorems by F. Browder [31] for solutions of elliptic PDEs. Theorems [31] can be readily modified to include homogeneous boundary conditions on some part of the boundary.

A Cartesian product $\mathcal{H}(\mathcal{D}, \mathcal{N}) \overset{\text{def}}{=} \overset{\circ}{W}{}_2^1(\mathcal{D}) \times L_2(\mathcal{N})$ consisting of ordered pairs $\{a_{\mathcal{D}}, a_{\mathcal{N}}\}$, $a_{\mathcal{D}} \in \overset{\circ}{W}{}_2^1(\mathcal{D})$, $a_{\mathcal{N}} \in L_2(\mathcal{N})$, is a Hilbert space with the inner product

$$\big[\{a_{\mathcal{D}}, a_{\mathcal{N}}\}, \{b_{\mathcal{D}}, b_{\mathcal{N}}\}\big]_{\mathcal{H}} = \int_{\mathcal{D}} a_{\mathcal{D}}\, b_{\mathcal{D}}\, \mathrm{d}s + \int_{\mathcal{D}} (\nabla_\Gamma a_{\mathcal{D}}, \nabla_\Gamma b_{\mathcal{D}})\, \mathrm{d}s + \int_{\mathcal{N}} a_{\mathcal{N}}\, b_{\mathcal{N}}\, \mathrm{d}s$$

which induces the norm

$$\|a_{\mathcal{D}}, a_{\mathcal{N}}\|^2_{\mathcal{H}} = \int_{\mathcal{D}} |a_{\mathcal{D}}|^2\, \mathrm{d}s + \int_{\mathcal{D}} |\nabla_\Gamma a_{\mathcal{D}}|^2\, \mathrm{d}s + \int_{\mathcal{N}} |a_{\mathcal{N}}|^2\, \mathrm{d}s.$$

Let $L : H(\Gamma) \to \mathcal{H}(\mathcal{D}, \mathcal{N})$ be a linear operator defined as

$$L\psi = \left\{ \psi\Big|_{\mathcal{D}}, \frac{\partial\psi}{\partial\nu}\Big|_{\mathcal{N}} \right\} \quad \forall \psi \in H(\Gamma).$$

From Theorem 3 follows

Corollary 1. *The linear operator L is an isometry of $H(\Gamma)$ onto $\mathcal{H}(\mathcal{D}, \mathcal{N})$.*

For the basic functions $\{\Psi_k\}$, from Corollary 1 and Theorem 4 follows

Corollary 2. *The system $\{L\Psi_k\}$ is complete and minimal in $\mathcal{H}(\mathcal{D}, \mathcal{N})$.*

Applying Corollary 2, we approximate the solution $\psi(r, \theta, \varphi)$ of the BVP (4)–(6) by a sequence $\{\psi^{(N)}(r, \theta, \varphi)\}$ of linear combinations with respect to the first N basic functions

$$\Psi_k : \psi(r, \theta, \varphi) = \lim_{N\to\infty} \psi^{(N)}(r, \theta, \varphi), \qquad \psi^{(N)}(r, \theta, \varphi) = \sum_{k=1}^{N} Q_k^{(N)}\, \Psi_k(r, \theta, \varphi). \tag{23}$$

Here coefficients $Q_k^{(N)}$ are to be found using the condition of the least square deviation of the approximate solution $\psi^{(N)}$ from the boundary function $h = \{h_{\mathcal{D}}, h_{\mathcal{N}}\} \in \mathcal{H}(\mathcal{D}, \mathcal{N})$ corresponding to (6) on Γ: $\| L\psi^{(N)} - h\|_{\mathcal{H}} \to \min$. This condition leads to the following system of linear equations with respect to the unknown coefficients $Q_k^{(N)}$:

$$\sum_{k=1}^{N} Q_k^{(N)} G_k^l = h^l , \quad l = 1, 2, \ldots, N,$$

with elements of this linear system defined as

$$G_k^l = [L\Psi_k, L\Psi_l]_{\mathcal{H}}, \qquad h^l = [h, L\Psi_l]_{\mathcal{H}}.$$

The method of least squares guarantees the convergence of the sequence $L\psi^{(N)}$ in the Hilbert space $\mathcal{H}(\mathcal{D}, \mathcal{N})$, whence, by Corollary 1, follows the convergence of the sequence $\psi^{(N)}$ in the Hilbert space $H(\Gamma)$. Now for the sequence of approximate solutions $\{\psi^{(N)}\}$, reference to Theorem 3 completes the proof of its convergence in $W_2^{3/2}(\Omega)$ to the exact solution $\psi \in \overset{\circ}{W}{}_2^1(\Omega, \gamma) \cap W_2^{3/2}(\Omega)$.

4.3. Asymptotics near the edges

Turn again to the selected edge mentioned in Sec. 4.1. Introduce a cylindrical system of co-ordinates related to this edge by the use of the Cartesian X, Y, Z and the spherical (r, Θ, Φ) co-ordinate systems defined in Sec. 4.1. Namely, let Z be the coordinate from the above Cartesian system, Φ the co-ordinate from the above spherical system, and ρ be defined by the formula $\rho = \sqrt{r^2 - Z^2}$. Then the desired cylindrical co-ordinate system is (ρ, Z, Φ).

Starting from the view (23) of the solution and using representation (22) for the Multipoles, we derive an asymptotics for the solution of the BVP near the edge with dihedral angle of value $\pi\beta$

$$\Psi \sim \rho^{1/\beta} \sin\frac{\Phi}{\beta} [\mathcal{J}_{1,1}\, Z^{\mu_1 - 1/\beta} + \ldots] + \rho^{2/\beta} \sin\frac{2\Phi}{\beta} [\mathcal{J}_{2,1}\, Z^{\mu_2 - 1/\beta} + \ldots] + \ldots, \qquad \rho \to 0,\ Z \to 0. \quad (24)$$

Quantities $\mathcal{J}_{1,1}$ and $\mathcal{J}_{2,1}$ appearing here can be expressed via coefficients of expansions (22), (23), in particular

$$\mathcal{J}_{1,1} = 2^{-1/\beta} [\Gamma(1 + 1/\beta)]^{-1} Q_1 D_1^1,$$

where $\Gamma(x)$ is Gamma-function [30].

Note that coefficients Q_k^n in expansion (23) are named intensity factors at the vertex of the cone (polyhedral angle) and quantities $\mathcal{J}_{1,1}$, $\mathcal{J}_{2,1}$ the intensity factors at its edge. From what was said it follows that our method provides computation of all mentioned intensity factors along with the solution itself.

4.4. Numerical solution of BVPs for "Cube in Cube" and the Fichera corner

Let Cartesian co-ordinates of a point y in $\mathbf{R}^3$ be (y_1, y_2, y_3). Denote by $\mathcal{G}_{2s}$ a cube with edge length $2s > 0$: $\mathcal{G}_{2s} = \{(y_1, y_2, y_3) : y_i \in (-s, s)\}$. Let $\mathcal{G} = \mathcal{G}_4 \setminus \overline{\mathcal{G}_2}$ be a domain, which is a cube of edge length 4 with cut out cube of edge length 2. We call this domain (shown in Fig. 8) "Cube in Cube", and pose in it the following Dirichlet problem:

$$\Delta\psi = 0 \quad \text{in } \mathcal{G}, \qquad \psi = 0 \quad \text{on } \partial\mathcal{G}_2, \qquad \psi = 1 \quad \text{on } \partial\mathcal{G}_4. \quad (25)$$

This BVP is related, in particular, to computation of the electrical capacity $\mathbf{C}$ of a condenser, whose plates are $\partial\mathcal{G}_2$ and $\partial\mathcal{G}_4$. Quantity $\mathbf{C}$ is expressed by means of the solution of (25) as follows:

$$\mathbf{C} = \frac{1}{4\pi} \int_{\mathcal{G}} |\text{grad}\, \psi|^2 \, dw, \quad (26)$$

where dw is an element of volume.

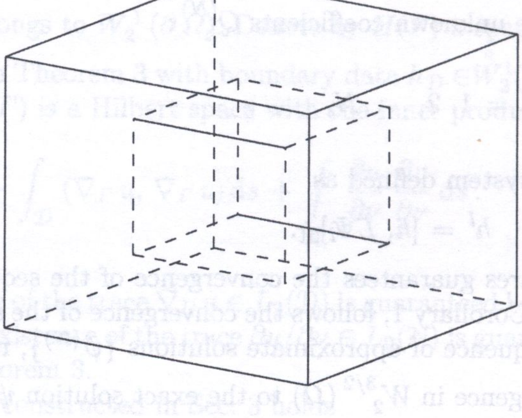

Fig. 8

Observe that domain $\mathcal{G}$ has a symmetry with respect to each of three planes $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$. If we split this domain along those planes, we find that it is broken down into eight pieces, being congruent each to other. Let us denote one of them by $\Omega = \{(y_1, y_2, y_3) : y_i \in (0, 2)\} \setminus \{(y_1, y_2, y_3) : y_i \in [0, 1]\}$.

This domain Ω (shown in Fig. 9) is known as "the Fichera corner" [2]. Its boundary can be presented as a union of three parts, $\partial\Omega = \gamma \cup \mathcal{D} \cup \mathcal{N}$. Here γ and $\mathcal{D}$ are common parts of $\partial\Omega$ with $\partial\mathcal{G}_2$ and $\partial\mathcal{G}_4$, respectively, and $\mathcal{N}$ is the rest of it.

Just defined domain Ω is a particular case of the domain with the same designation introduced in Sec. 2. Here, the roles of surfaces γ, $\mathcal{D}$ and $\mathcal{N}$ are played by the surfaces with the same designations. The role of a cone $\mathbf{K}$ is played by the trihedral corner $\mathbf{K}^\alpha$ with $\alpha = 1/2$; obviously $\mathbf{K}^\alpha$ is an extension of Ω through $\Gamma = \mathcal{D} \cup \mathcal{N}$ likewise to Sec. 2; γ is also a closure of a simply connected domain on $\partial\mathbf{K}^\alpha$, containing its vertex as an interior point.

In order to reduce this geometry to definitions accepted in Sec. 2.1, let us replace the origin of co-ordinates at the vertex of the trihedral corner, dispose axis x_3 so that it is inclined at equal angles (more then $\pi/2$) to each of edge of the trihedral corner, and introduce the spherical co-ordinates (r, θ, φ) so that angle θ is reckoned from x_3. Then contour $\mathcal{L}$ presents $\mathcal{L}^\alpha$ described by the relations (1), (2). It is evident that BVP (4) in $\mathcal{G}$ is equivalent to BVP (4), (5) in the Fichera corner Ω with $h_{\mathcal{D}} \equiv 1$ and $h_{\mathcal{N}} \equiv 0$.

The method related in Sec. 4.2 was applied to this BVP. Note that in the sum (23) the representation of the Multipoles will be more convenient in the form (20) than (19). Taking into account the symmetry of the solution Ψ, we emphasize that nonzero coefficients in the sum (23) will have Multi-

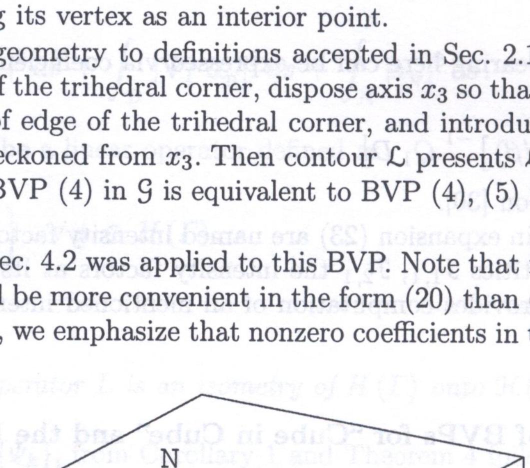

Fig. 9

poles $\Psi_n^{m\pm}$ only with first upper index divisible by 3 and second upper index "+". For instance, the representation (23) for an approximate solution $\Psi^{(9)}$ containing only nonzero terms has the form:

$$\Psi^{(9)} = Q_1^{(9)}\Psi_1^{0+} + Q_2^{(9)}\Psi_2^{0+} + Q_3^{(9)}\Psi_1^{3+} + Q_4^{(9)}\Psi_3^{0+} + Q_5^{(9)}\Psi_2^{3+} + Q_6^{(9)}\Psi_4^{0+} + Q_7^{(9)}\Psi_3^{3+} + Q_8^{(9)}\Psi_5^{0+} + Q_9^{(9)}\Psi_1^{6+} . \quad (27)$$

Numerical experiments proved that the method possesses high effectiveness. It has exponential rate of its convergence and ensures precise computation of the solution and its derivatives up to the surface of the trihedral corner by virtue of only few numbers of degrees of freedom. In particular, by means of only 9 Multipoles Ψ_k like in (27) we obtained the solution Ψ for the considered problem and its gradient with accuracy 10^{-4} or better near the surface of the trihedral corner. This result is well illustrated by Fig. 10, where the level surface $\psi = 0.1$ is presented. Solid lines correspond to edges of the cube $\mathcal{G}_2$.

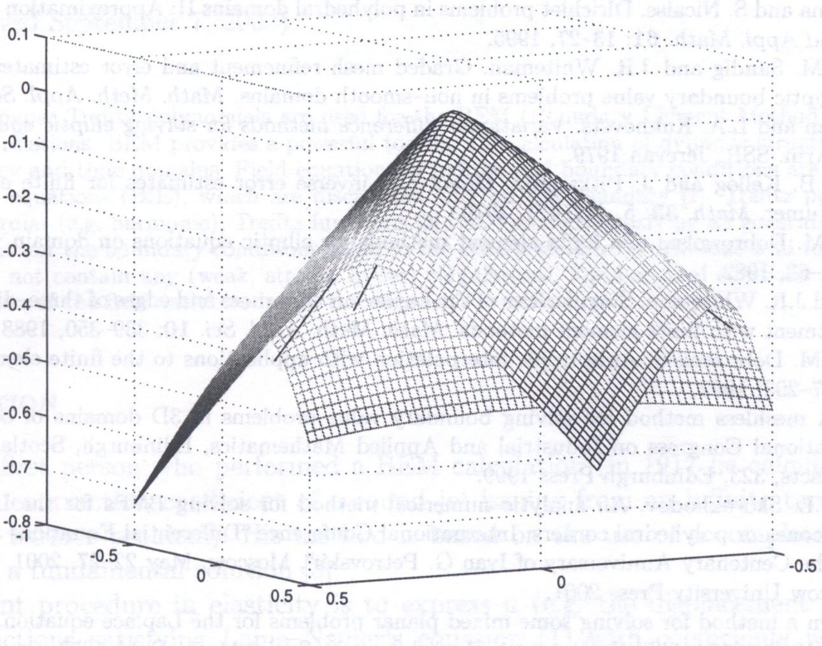

Fig. 10

Besides, a value of capacity (26) of "Cube in Cube" was computed: $\mathbf{C} = 2.6696$. It is worth to note that this value is 7.5% greater than the capacity of a condenser of two concentric balls with the same volumes as $\mathcal{G}_4$ and $\mathcal{G}_2$.

In addition, intensity coefficient in formula (24) was computed: $\mathfrak{I}_{1,1} = 0.987741$. So, asymptotic relation (24) for this solution near an edge of the trihedral corner looks like

$$\Psi \sim \rho^{2/3} \sin\frac{2\Phi}{3}\,[0.987741\,Z^{-0.213121} + \ldots] + \ldots .$$

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